Reconnection of Unstable/Stable Manifolds of the Harper Map

— *Asymptotics-Beyond-All-Orders Approach* —

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The Harper map is one of the simplest chaotic systems exhibiting reconnection of invariant manifolds. The method of asymptotics beyond all orders (ABAO) is used to construct unstable/stable manifolds of the Harper map. By enlarging the neighborhood of a singularity, the perturbative solution of the unstable manifold is expressed as a Borel summable asymptotic expansion in a sector including $t = -\infty$ and is analytically continued to the other sectors, where the solution acquires new terms describing heteroclinic tangles. When the parameter changes to the reconnection threshold, the unstable/stable manifolds are shown to acquire new oscillatory portion corresponding to the heteroclinic tangle after the reconnection.

§1. Introduction

Bifurcation involving chaotic motion is one of the origins of diversity in complex systems and reconnection among unstable/stable manifolds is such an example. However, general features of the interplay between bifurcation and chaos are not well understood and a case study on a simple system would provide useful information for the generic behavior. Moreover, since the trajectory near the bifurcation point is considered to be very complicated, an analytical approach would be more helpful. Thus, as one of the simplest systems exhibiting reconnection,[1),2)] we analytically study the nearly integrable Harper map.

The Harper map depends on a real parameter k and is defined on the torus $\{(v,u) \in [-\pi,\pi]^2\}$:

$$\begin{aligned} v(t+\sigma) - v(t) &= -\sigma \sin u(t) \\ u(t+\sigma) - u(t) &= k\sigma \sin v(t+\sigma) \end{aligned} \tag{1·1}$$

where $\sigma(>0)$ is the time step and it plays a role of the small parameter. Since the case of $k<0$ is conjugate to that of $k>0$[2)] and since the solution for $k>1$ can be obtained from that for $k<1$ by a simple symmetry argument (cf. Appendix A), it is sufficient to consider the case of $0<k\le 1$. In the continuous time limit, where $\sigma \to 0$, the map reduces to an integrable set of differential equations:

$$\begin{aligned} v'(t) &= -\sin u(t) \\ u'(t) &= k \sin v(t) \ . \end{aligned} \tag{1·2}$$

Throughout this paper, the prime is used to indicate the differentiation with respect

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to time t such as $u'(t) \equiv \dfrac{du(t)}{dt}$. Eq. (1·2) admits topologically different separatrices depending on the parameter k (cf. Fig. 1) and the separatrix changes its shape when $k \to 1$. As shown in the figure, when k increases, the middle point of the upper separatrix moves upward and, when $k = 1$, it reaches the fixed point $(0, \pi)$. This view would be changed for nonvanishing σ due to the heteroclinic tangles of stable and unstable manifolds. Indeed, in case of $0 < k < 1$, the unstable manifold of the fixed point $(\pi, 0)$ is expected to show an oscillatory behavior near $(-\pi, 0)$, while in case of $k = 1$, it would oscillate near $(-\pi, 0)$ *and* $(0, \pi)$. We, then, study the crossover between the two behaviors based on an analytical solution.

One of the powerful methods analytically dealing with heteroclinic/homoclinic tangles of invariant manifolds is the method of asymptotics beyond all orders (ABAO method). The key idea is to employ the so-called inner equation and to investigate it with the aid of the resurgence theory[3)] and the Borel resummation. The inner equation magnifies the behavior of the solution near its singularities and bridges the solutions in different sectors.

The idea of the inner equation was first used by Lazutkin[4)] to derive the first crossing angle between the stable and unstable manifolds of Chirikov's standard map, and by Kruskal and Segur,[5)] to study a singular perturbation problem of ordinary differential equations. Lazutkin and his coworkers[6),7),8),9)] developed their method, and Lazutkin, Gelfreich and Svanidze[10)] proposed a method of systematically improving the solution of the inner equation for the standard map. The first rigorous proof of the method was given by Gelfreich.[11),12)] He and his collaborators[16),15),13),14)] have also studied the splitting of separatrices and related bifurcations for various systems. Meanwhile, Hakim and Mallick[17)] used the technique of the Borel transformation within this context and obtained the first crossing angle previously obtained in Ref. 4). Tovbis, Tsuchiya and Jaffé[18)] improved their method and derived analytical approximations of perturbed unstable/stable manifolds for the Hénon map. Nakamura and Hamada[19),20)] studied the separatrix splitting for the cubic map and, as in the work of Ref. 10), Nakamura and Kushibe[21)] considered the systematic improvement of the solution of the standard map. Also, ABAO method was applied to some higher dimensional systems by Hirata, Nozaki and Konishi.[22),23)]

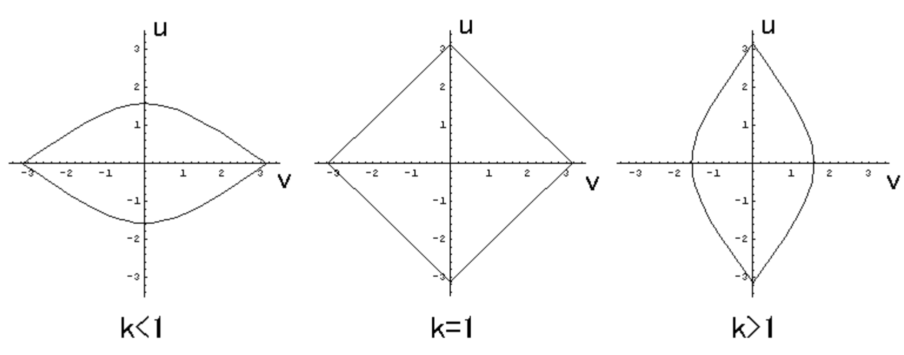

Fig. 1. Separatrix of the Harper map in the continuous limit.

Roughly speaking, the procedure obtaining an approximate expression of the unstable manifold can be summarized as follows:

(i) Find singularities in the complex time domain of the lowest order perturbative solution.

(ii) Magnify the neighborhood of a singularity closest to the real axis and derive an asymptotic expansion of the lowest order solution valid in a certain sector.

(iii) The Borel transformation and the resurgence theory are used to construct the asymptotic expansion which is valid in the other sector. Usually, there appear additional terms which are exponentially small for real time values.

(iv) Going back to the original equation, derive equations for the exponentially small corrections corresponding to the terms found in the previous step. An appropriate solution is chosen by matching its asymptotic expansion with that obtained in the previous step.

In this paper, with the aid of the ABAO method, the reconnection of the unstable manifold is studied for the Harper map. Our presentation mainly follows the work of Nakamura and Kushibe.[21] In the text, an approximate expression of the unstable manifold is constructed and only the result is given for the stable manifold.

The rest of this paper is arranged as follows: In the next section, an outline of our result is presented without proofs and the details, which are given in Secs. 3-6. In Sec. 3, a perturbative solution is constructed by the Melnikov perturbation method and its asymptotic expansion is derived. In Sec. 4, the resurgence theory and the Borel resummation method are applied to solve the inner equation, and asymptotic expansions of the additional terms are derived. In Sec. 5, the exponentially small corrections are obtained by matching the solutions of the original and inner equations. As will be discussed later, the separatrix solution of (1·2) has singularities along two lines both parallel to the imaginary axis and, among them two pairs on individual lines mainly contribute to the heteroclinic tangles of the manifolds. Secs. 4 and 5 deal with one pair of singularities which have smaller real part and, in Sec. 6, the same analysis is repeated for the other pair of singularities. In Sec. 7, the derived solution is compared with numerical calculations and the reconnection of the unstable manifold is discussed. Sec. 8 is devoted to summary.

§2. Outline of the Construction of Unstable Manifold

In this section, we summarize the construction of the unstable manifolds of the Harper map with the aid of the method of asymptotics beyond all orders (ABAO method). The details of the analysis will be described in Secs. 3-6.

2.1. *Unstable manifold*

We are interested in the unstable manifold of the Harper map satisfying the boundary condition:

$$v(-\infty) = \pi, \quad u(-\infty) = 0 \tag{2·1}$$

When the parameter σ is small enough, the solution of (1·2) with the same boundary condition provides a good approximation for $t \ll 0$. However, the solution is not

valid for the whole real t since exponentially small terms should be added when t exceeds certain values $t = T_1, T_2$ ($T_1 < T_2$). In fact, the unstable manifold is well-approximated by a double expansion with respect to σ and $\epsilon \equiv \sigma^j e^{-\frac{c}{\sigma}}$ ($c > 0, j$: some constant):

$$\begin{pmatrix} v(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} v_0(t,\sigma) \\ u_0(t,\sigma) \end{pmatrix} + S(t-T_1)\sum_{n=1}^{\infty}\begin{pmatrix} \mathrm{Re}\left(E_1(t)^n v_n^{(1)}(t,\sigma)\right) \\ \mathrm{Re}\left(E_1(t)^n u_n^{(1)}(t,\sigma)\right) \end{pmatrix}$$
$$+S(t-T_2)\sum_{n=1}^{\infty}\begin{pmatrix} \mathrm{Re}\left(E_2(t)^n v_n^{(2)}(t,\sigma,\epsilon)\right) \\ \mathrm{Re}\left(E_2(t)^n u_n^{(2)}(t,\sigma,\epsilon)\right) \end{pmatrix} \qquad (2\cdot2)$$

where $S(t)$ denotes the step function, $v_n^{(1)}(t,\sigma)$, $u_n^{(1)}(t,\sigma)$, $v_n^{(2)}(t,\sigma,\epsilon)$ and $u_n^{(2)}(t,\sigma,\epsilon)$ admit power series expansions with respect to σ, and $E_l(t)$ is an exponentially small but highly oscillatory function of t (more precisely, $E_l(t) = \mathrm{O}(\epsilon)$ and is periodic with period σ).

The difference equations for $v_n^{(l)}, u_n^{(l)}$ are obtained by substituting (2·2) into (1.1) and equating the terms with the same powers in $E_l(t)$. Particularly, $v_1^{(l)}, u_1^{(l)}$ obey a linear equation and their values at $t = T_l$ are necessary for fixing them. Such boundary values are provided with the aid of the Borel summable asymptotic expansions valid near a singularities in the complex t-plane. To see the method more in detail, let us focus on the time domains $-\infty < t < T_1$ and $T_1 < t < T_2$.

2.2. *Melnikov perturbation*

The solution $v_0(t), u_0(t)$ describing the unstable manifold in the domain $-\infty < t < T_1$ is given by the conventional Melnikov perturbation:

$$v_0(t,\sigma) \equiv \sum_{i=0}^{\infty}\sigma^i v_{0i}(t), \quad u_0(t,\sigma) \equiv \sum_{i=0}^{\infty}\sigma^i u_{0i}(t)$$

The lowest order solutions $u_{00}(t), v_{00}(t)$ are nothing but the solutions of the differential equation (1·2) with the boundary condition (2·1). After fixing the origin of t by $v_0(T) = \mathrm{O}(\sigma^2)$, the terms $(v_{00}(t), u_{00}(t))$ and $(v_{0i}(t), u_{0i}(t))$ ($i \geq 1$) are found to have, respectively, logarithmic branch points and poles at

$$t = \frac{1}{\sqrt{k}}\left(n+\frac{1}{2}\right)\pi i, \quad 2T + \frac{1}{\sqrt{k}}\left(n+\frac{1}{2}\right)\pi i \qquad (n = 0, \pm1, \pm2, \cdots) \qquad (2\cdot3)$$

where $T = \frac{1}{\sqrt{k}}\ln\frac{1+\sqrt{k}}{\sqrt{1-k}} > 0$. On the other hand, by fixing the time origin as $v_0(0) = \pi/2 + \mathrm{O}(\sigma^2)$, the perturbative solution of the unstable manifold for $k = 1$ have singularities at

$$t = \left(n+\frac{1}{2}\right)\pi i$$

As one can easily seen from the definition of T, T diverges as $k \to 1-0$. We choose the time origin so that the solution converges as $k \to 1-0$ and the location of the singularities in the limit of $k \to 1-0$ matches with that for the case of $k=1$. We remark that the choice of the time origin, $v_0(T)=0$ *), gives the solution which does not converge as $k \to 1-0$ (see appendix E).

In the domain Re $t<0$ of the complex time plane, $(v_0(t), u_0(t))$ is a reasonable approximation of the unstable manifold since individual terms of their expansions have no singularities there. The expressions valid in the domain Re $t>0$ can be constructed by analytically continuing v_0, u_0 along any curve which does not cross the lines $t=\pm is$ $(s>\frac{\pi}{2\sqrt{k}})$ emanating from the singularities of v_0, u_0. Then, it is convenient to study the analytic continuation along the curve passing near the singular points since the divergence of the perturbative solutions is expected to magnify the small corrections.

2.3. *Behavior near a singularity*

Now we focus on the behavior of the perturbative solution near a singular point $t=t_1 \equiv \frac{i\pi}{2\sqrt{k}}$. The corrections v_{0i}, u_{0i} have the Laurent expansions:

$$\sigma^i \begin{pmatrix} v_{0i}(t) \\ u_{0i}(t) \end{pmatrix} = \frac{\sigma^i}{(t-t_1)^i} \sum_{l=0}^{\infty} \begin{pmatrix} a_l^{(i)}(t-t_1)^l \\ b_l^{(i)}(t-t_1)^l \end{pmatrix} , \quad (i \geq 1, b_0^{(i)} \neq 0) . \tag{2·4}$$

Hence, higher order corrections are more singular. In such a case, the behavior of the most divergent term is important. Because the most divergent contribution, e.g., to v_0 has the form:$\sum_i \sigma^i a_l^{(i)}/(t-t_1)^i$, it is convenient to introduce a new variable $z=\frac{t-t_1}{\sigma}$ and to rearrange the expansions as double expansions with respect to σ and z. Since one has

$$\begin{aligned} \sigma^1 v_{01}(t,\sigma) &= \frac{a_0^{(1)}}{z} + \sigma \frac{a_1^{(1)}}{z^0} + \cdots \\ \sigma^2 v_{02}(t,\sigma) &= \frac{a_0^{(2)}}{z^2} + \sigma \frac{a_1^{(2)}}{z^1} + \cdots \\ &\cdots\cdots\cdots\cdots\cdots \\ \sigma^i v_{0i}(t,\sigma) &= \frac{a_0^{(i)}}{z^i} + \sigma \frac{a_1^{(i)}}{z^{i-1}} + \cdots \end{aligned}$$

and the similar expression for u_{0i}, the perturbative solution can be reexpressed as

$$\begin{pmatrix} v_0(t,\sigma) \\ u_0(t,\sigma) \end{pmatrix} = \sum_{i=0}^{\infty} \sigma^i \begin{pmatrix} v_{0i}(t) \\ u_{0i}(t) \end{pmatrix} = \begin{pmatrix} v_{00}(t) \\ u_{00}(t) \end{pmatrix} + \sum_{l=0}^{\infty} \sigma^l \begin{pmatrix} V_{0l}(z) \\ U_{0l}(z) \end{pmatrix} \tag{2·5}$$

where V_{0l}, U_{0l} are given by

$$V_{0l}(z) = \sum_{i=1}^{\infty} \frac{a_l^{(i)}}{z^{i-l}} , \qquad U_{0l}(z) = \sum_{i=1}^{\infty} \frac{b_l^{(i)}}{z^{i-l}} \tag{2·6}$$

*) This choice gives the symmetric unstable manifold with respect to u axis. But it makes Stokes multiplier divergent and it indicates that the unstable manifold is not symmetric with respect to u axis even within the perturabation approach.

We remark that these series are divergent and are meaningful only as asymptotic expansions.

In short, the perturbative solution valid in the domain $-\infty < \mathrm{Re}\, t < 0$ turns out to be an asymptotic expansion near its singularity. Therefore, a careful treatment is necessary to analytically continue the solution across the line Re $t = 0$.

2.4. *Borel resummation and analytic continuation*

Fortunately, the series (2·6) are Borel summable. This can be shown as follows:

(i) The equations for the Borel transforms $\widetilde{V}_{0l}(p), \widetilde{U}_{0l}(p)$ of $V_{0l}(z), U_{0l}(z)$ are derived from the original equation (1·1) and the expansion (2·5).

(ii) The singularities of the Borel transforms are found with the aid of the linearized inner equation. Indeed, according to the resurgence theory,[3),14)] the functional forms of the Borel transforms $\widetilde{V}_{0l}(p), \widetilde{U}_{0l}(p)$ are obtained from the linearized equation except some constants.

(iii) The Borel transformed equation is solved. This is carried out as follows: The previous step determines the functional forms of $\widetilde{V}_{0l}(p), \widetilde{U}_{0l}(p)$ up to a few arbitrary constants. These constants are settled by comparing the power series expansions of the guessed solutions with the power series solutions of the Borel transformed equation, which are numerically derived.

Then, the Borel transforms $\widetilde{V}_{0l}(p), \widetilde{U}_{0l}(p)$ are found to have singularities along the lines $p = is,\ (s < -2\pi$ or $s > 2\pi)$ and are analytic near the origin. The analyticity implies the existence of the asymptotic expansions of $V_{0l}(z), U_{0l}(z)$ at $|z| \to \infty$ in a certain sector. Hence, the divergent series (2·6) are Borel summable.

Now we consider the analytic continuations of $V_{0l}(z), U_{0l}(z)$. Since negative powers of z in V_{0l}, U_{ol} are expressed as the inverse Borel transformation (namely the Laplace transformation) of functions $\widetilde{V}_{0l}(p), \widetilde{U}_{0l}(p)$ which are analytic except on the lines $p = is,\ (s < -2\pi$ or $s > 2\pi)$, their analytic continuations from the sectors Re $z < 0$, Im $z < 0$ to Re$z > 0$, Im$z < 0$, are obtained by appropriately changing the integration path of the Borel transformation. Then, in the sector $0 <$ Re z, Im $z < 0$, the solutions are expressed as

$$\begin{pmatrix} v_{00}(t) \\ u_{00}(t) \end{pmatrix} + \sum_{l=0}^{\infty} \sigma^l \begin{pmatrix} V_{0l}(z) \\ U_{0l}(z) \end{pmatrix} + e^{-2\pi i z} \sum_{l=0}^{\infty} \sigma^l \begin{pmatrix} V_{1l}(z) \\ U_{1l}(z) \end{pmatrix} + \cdots \tag{2·7}$$

where $V_{1l}(z), U_{1l}(z)$ are additional contributions and only the terms dominant for real t are retained. More concretely, up to σ^0 and z^{-2}, one has

$$\begin{aligned} & e^{-2\pi i z} \sum_{l=0}^{\infty} \sigma^l \begin{pmatrix} V_{1l}(z) \\ U_{1l}(z) \end{pmatrix} \\ & = e^{-2\pi i z} \left[\Lambda_A \left\{ \begin{pmatrix} -\frac{1}{z} \\ \frac{1}{z} \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{2z^2} \end{pmatrix} \right\} + \Lambda_B \left\{ \begin{pmatrix} z \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{12z} \end{pmatrix} \right\} + \cdots \right] \end{aligned} \tag{2·8}$$

where

$$\Lambda_\alpha = \Lambda_0^\alpha + \sigma \Lambda_1^\alpha + \cdots \quad (\alpha = A, B)$$

and the coefficients Λ_i^α are numerically determined. Note that the analytic continuation (2·7) simply provides the asymptotic expansion near $t = t_1$ and that further investigation is necessary to construct the analytically continued solution.

2.5. *Solutions in different domains*

Now we consider the solution for Re $t > 0$. By comparing (2·2) with (2·8) and reminding $\sigma z = t - t_1$, one has

$$E_1(t) = \frac{e^{-\frac{\pi^2}{\sigma\sqrt{k}}}}{\sigma} e^{-\frac{2\pi i t}{\sigma}} , \tag{2·9}$$

and, thus, $\epsilon = e^{-\frac{\pi^2}{\sigma\sqrt{k}}}/\sigma$. Because $E_1(t)$ is exponentially small with respect to σ and periodic in t with period σ, equations for $(v_n^{(1)}, u_n^{(1)})$ are obtained by substituting (2·2) to (1·1) and equating the terms with the same powers in $E_1(t)$. Substituting $v_n^{(1)}(t,\sigma) = \sum_{i=0}^\infty \sigma^i v_{ni}^{(1)}(t)$ and $u_n^{(1)}(t,\sigma) = \sum_{i=0}^\infty \sigma^i u_{ni}^{(1)}(t)$ into the so-obtained difference equations, one has a system of differential equations $(n,i) \neq (0,0)$:

$$\begin{cases} v_{ni}^{(1)\prime}(t) + \displaystyle\sum_{j=2}^{i} \left(\frac{d}{dt}\right)^j \frac{v_{n\ i-j+1}^{(1)}}{j!} = -u_{ni}^{(1)}(t)\cos u_{00}(t) + f_{ni}^u(t) \\ u_{ni}^{(1)\prime}(t) - \displaystyle\sum_{j=2}^{i} (-1)^j \left(\frac{d}{dt}\right)^j \frac{u_{n\ i-j+1}^{(1)}}{j!} = k v_{ni}^{(1)}(t)\cos v_{00}(t) + f_{ni}^v(t) \end{cases} \tag{2·10}$$

where $f_{ni}^v(t)$ and $f_{ni}^u(t)$ are functions of $v_{jk}^{(1)}$ $(j < n,\ k \le i)$, $v_{nk}^{(1)}$ $(k < i)$, v_{0k} $(k \le i)$ and $u_{jk}^{(1)}$ $(j < n,\ k \le i)$, $u_{nk}^{(1)}$ $(k < i)$, u_{0k} $(k \le i)$, respectively. The general solution of the system involves several constants of integration, which should be chosen so that the matching condition:

$$\sum_{i=0}^\infty \sigma^i \begin{pmatrix} v_{1i}^{(1)}(t_1 + \sigma z) \\ u_{1i}^{(1)}(t_1 + \sigma z) \end{pmatrix} = \sigma \sum_{l=0}^\infty \sigma^l \begin{pmatrix} V_{1l}(z) \\ U_{1l}(z) \end{pmatrix} \tag{2·11}$$

is satisfied for small σ and large z. Another matching condition at $t = t_1^*$ similar to (2·11) is obtained by repeating the procedures explained in the previous subsections. One can prove, from real analyticity of $(v_{0j},\ u_{0j})$, that the additional term obtained from the anlysis of $t = t_1^*$ is a conjugate to the term obtained from the analysis of $t = t_1$ (see appendix B). These two matching conditions fix the constants of integration and, thus, the solutions $v_n^{(1)}(t,\sigma), u_n^{(1)}(t,\sigma)$. As easily seen, the solutions have no singularity in $0 <$ Re $t < 2T$.

This completes the derivation of the solution valid in the domain $0 <$ Re $t < 2T$.

When Re t exceeds $2T$, new terms will appear and the solutions valid in the domain Re $t > 2T$ can be obtained as follows:

(i) The asymptotic expansions valid near $t = t_2 \equiv 2T + i\frac{\pi}{2\sqrt{k}}$ are derived as a function of $z' = (t - t_2)/\sigma$ and are analytically continued from the domain Re$z' < 0$, Im$z' < 0$ to Re$z' > 0$, Im$z' < 0$ with the aid of the Borel resummation and the resurgence theory.

(ii) The system of differential equations for the additional terms in Re $t > 2T$ is derived from (1·1) and (2·2). And its general solutions are calculated.
(iii) From the matching conditions such as (2·11) near the singularities $t = t_2, t_2^*$, the constants of integration are fixed and one gets the desired solutions.

The final solution is shown to be analytic in the domain Re $t > 2T$ and, thus, the description of the unstable manifold on the whole time domain is completed. Note that, when t passes the line Re $t = 2T$, the terms added in the domain $0 <$ Re $t < 2T$ generate new terms, which are of order of ϵ^2 and negligible for real t. It will be proved in Sec. 6.

2.6. *Approximate unstable manifold*

Up to the order of σ^3 and $\sigma^1\epsilon$, the unstable manifold is expressed as

$$\begin{aligned}
v_u(t) =& \, v_{00}(t) + \sigma^2 v_{02}(t) \\
&+ S(t)\mathrm{Re}\left[\frac{4\Lambda^{(1)}}{i\sigma} e^{\frac{2\pi i t_1}{\sigma}} x_2(t) e^{-\frac{2\pi i t}{\sigma}}\right] \\
&+ S(t-2T)\mathrm{Re}\left[\frac{4\Lambda^{(2)}}{i\sigma} e^{\frac{2\pi i t_2}{\sigma}} \left\{\left(\frac{T(k-1)^2-(1+k)}{4k(k-1)}\right) x_1(t) + x_2(t)\right\} e^{-\frac{2\pi i t}{\sigma}}\right] \\
u_u(t) =& \, u_{00}(t) + \sigma\frac{y_1(t)}{2} + \sigma^2 u_{02}(t) + \sigma^3\left(\frac{1}{2}u_{02}'(t) - \frac{1}{24}y_1''(t)\right) \\
&+ S(t)\mathrm{Re}\left[\frac{4\Lambda^{(1)}}{i\sigma} e^{\frac{2\pi i t_1}{\sigma}} \left(y_2(t) + \sigma\frac{y_2'(t)}{2}\right) e^{-\frac{2\pi i t}{\sigma}}\right] \\
&+ S(t-2T)\mathrm{Re}\Bigg[\frac{4\Lambda^{(2)}}{i\sigma} e^{\frac{2\pi i t_2}{\sigma}} \Bigg\{\left(\frac{T(k-1)^2-(1+k)}{4k(k-1)}\right)\left(y_1(t) + \sigma\frac{y_1'(t)}{2}\right) \\
&\qquad\qquad + \left(y_2(t) + \sigma\frac{y_2'(t)}{2}\right)\Bigg\} e^{-\frac{2\pi i t}{\sigma}}\Bigg]
\end{aligned}$$

where $S(t)$ stands for the step function and the other functions are defined as follows:

$$\begin{aligned}
v_{00}(t) &= -2\tan^{-1}\left[\sinh\sqrt{k}t - \sqrt{k}\cosh\sqrt{k}t\right] \\
u_{00}(t) &= 2\tan^{-1}\left[\frac{\sqrt{k}}{-\sqrt{k}\sinh\sqrt{k}t + \cosh\sqrt{k}t}\right]
\end{aligned} \tag{2·12}$$

$$\begin{aligned}
v_{02}(t) &= -\frac{1}{24}\left[x_1'(t) + x_1(t)\left\{kt - 2\sqrt{k}\frac{\left(\sinh\sqrt{k}t - \sqrt{k}\cosh\sqrt{k}t\right)\left(\cosh\sqrt{k}t - \sqrt{k}\sinh\sqrt{k}t\right)}{(1+k)\cosh^2\sqrt{k}t - 2\sqrt{k}\sinh\sqrt{k}t\cosh\sqrt{k}t}\right\}\right] \\
u_{02}(t) &= \frac{1}{24}\left[2y_1'(t) - y_1(t)\left\{kt - 2\sqrt{k}\frac{\left(\sinh\sqrt{k}t - \sqrt{k}\cosh\sqrt{k}t\right)\left(\cosh\sqrt{k}t - \sqrt{k}\sinh\sqrt{k}t\right)}{(1+k)\cosh^2\sqrt{k}t - 2\sqrt{k}\sinh\sqrt{k}t\cosh\sqrt{k}t}\right\}\right]
\end{aligned} \tag{2·13}$$

$$
\begin{aligned}
x_1(t) &= -2\sqrt{k}\frac{-\sqrt{k}\sinh\sqrt{k}t+\cosh\sqrt{k}t}{1+\left(\sinh\sqrt{k}t-\sqrt{k}\cosh\sqrt{k}t\right)^2}\\
y_1(t) &= -2k\frac{\sinh\sqrt{k}t-\sqrt{k}\cosh\sqrt{k}t}{1+\left(\sinh\sqrt{k}t-\sqrt{k}\cosh\sqrt{k}t\right)^2}
\end{aligned}
\qquad (2\cdot14)
$$

$$
\begin{aligned}
x_2(t) &= -\frac{1}{4k}\left(\sinh\sqrt{k}t-\sqrt{k}\cosh\sqrt{k}t\right)+\frac{1+k}{4\sqrt{k}}\frac{\cosh\sqrt{k}t}{1+\left(\sinh\sqrt{k}t-\sqrt{k}\cosh\sqrt{k}t\right)^2}\\
&\qquad +\frac{1-k}{8k}\left(t-\frac{i\pi}{2\sqrt{k}}\right)x_1(t)\\
y_2(t) &= \frac{1}{4\sqrt{k}}\left(\cosh\sqrt{k}t-\sqrt{k}\sinh\sqrt{k}t\right)+\frac{1+k}{4\sqrt{k}}\frac{\cosh\sqrt{k}t}{1+\left(\sinh\sqrt{k}t-\sqrt{k}\cosh\sqrt{k}t\right)^2}\\
&\qquad +\frac{1-k}{8k}\left(t-\frac{i\pi}{2\sqrt{k}}\right)y_1(t)
\end{aligned}
\qquad (2\cdot15)
$$

The coefficients $\Lambda^{(1)}$, $\Lambda^{(2)}$ are numerically determined as

$$
\begin{aligned}
\Lambda^{(1)} &= i4\pi^3A_1+\sigma\pi^3\left\{-(k-1)B_2-\frac{kt_1+1}{12}B_4\right\}\\
\Lambda^{(2)} &= -i4\pi^3A_1+\sigma\pi^3\left\{-(k-1)B_2+\frac{kt_2-1}{12}B_4\right\}
\end{aligned}
\qquad (2\cdot16)
$$

where $A_1=0.27893,\ B_2=0.14,\ B_4=3.503.$

§3. Melnikov Perturbation

In this section, we start with the investigation of the perturbative solution, which provides a good approximation of the unstable manifold for $t\to-\infty$. The lowest order terms in the expansion $v_0(t,\sigma)\equiv\sum_{i=0}^{\infty}\sigma^i v_{0i}(t)$, $u_0(t,\sigma)\equiv\sum_{i=0}^{\infty}\sigma^i u_{0i}(t)$ satisfy (1·2) and the first and second order terms obey

$$
\begin{aligned}
v'_{01}(t)+u_{01}(t)\cos u_{00}(t) &= -\frac{1}{2}v''_{00}(t)\\
u'_{01}(t)-kv_{01}(t)\cos v_{00}(t) &= \frac{1}{2}u''_{00}(t)
\end{aligned}
\qquad (3\cdot1)
$$

$$
\begin{aligned}
v'_{02}(t)+u_{02}(t)\cos u_{00}(t) &= -\frac{1}{2}v''_{01}(t)-\frac{1}{6}v'''_{00}(t)+\frac{u_{01}^2(t)}{2}\sin u_{00}(t)\\
u'_{02}(t)-kv_{02}(t)\cos v_{00}(t) &= \frac{1}{2}u''_{01}(t)-\frac{1}{6}u'''_{00}(t)-k\frac{v_{01}^2(t)}{2}\sin v_{00}(t)
\end{aligned}
\qquad (3\cdot2)
$$

The lowest order solution of the unstable manifold with $v_{00}(T)=0$ is given by (2·12). Because of the boundary conditions $v_{0n}(t)\to0,\ u_{0n}(t)\to0$ for $t\to-\infty$ and

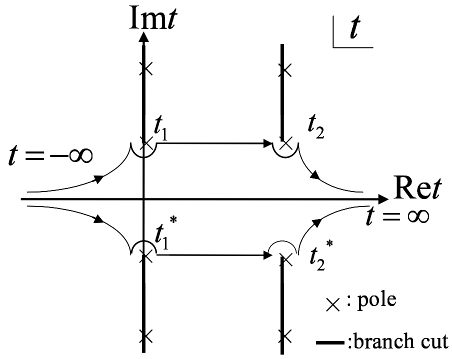

Fig. 2. Singular points (crosses) of the lowest order solution $v_0,\ u_0$ in the complex time domain.

$v_{01}(T) = v_{03}(T) = 0,\ v_{02}(T) = \frac{k^{3/2}}{12}\sqrt{1-k}$, the first and the third order solutions are found to be

$$v_{01}(t) = 0\ , \qquad u_{01}(t) = \frac{1}{2}y_1(t)$$

$$v_{03}(t) = 0\ , \qquad u_{03}(t) = \frac{1}{2}u'_{02}(t) - \frac{1}{24}y''_1(t)$$

and the second order solution is given by (2·13). It will be discussed in appendix E, we fix a time origin so that the final solution has no divergent term and the choice forbids a symmetric perturbative solution with respect to u axis. We observe that each order solution has two sequences of singular points in the complex t-plane (see Fig. 3).

$$t = \frac{1}{\sqrt{k}}\left(n+\frac{1}{2}\right)\pi i, \quad 2T + \frac{1}{\sqrt{k}}\left(n+\frac{1}{2}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \cdots) \tag{3·3}$$

where $T = \frac{1}{\sqrt{k}}\ln\frac{1+\sqrt{k}}{\sqrt{1-k}} > 0$. This implies that the perturbative solution approximates the unstable manifold well in the domain: Re $t < 0$.

In the rest of this section, we examine the behavior of the perturbative solution near the singular point in the upper half plane: $t_1 \equiv \frac{i\pi}{2\sqrt{k}}$ closest to the real axis. By substituting the Laurent expansion of each perturbative term to the set of equations (2·10), we inductively get

$$\sigma^i v_{0i}(t) = \frac{\sigma^i a_0^{(i)}}{(t-t_1)^i} + \frac{\sigma^i a_1^{(i)}}{(t-t_1)^{i-1}} + \cdots, \quad (i \geq 1)$$

and, thus,

$$v_0(t,\sigma) - v_{00}(t) = \sum_{i=1}^{\infty}\sigma^i v_{0i}(t) = \sum_{i=1}^{\infty}\sum_{l=0}^{\infty}\frac{a_l^{(i)}}{(t-t_1)^{i-l}}\sigma^i \tag{3·4}$$

This indicates that higher order terms with respect to σ have higher order poles at $t = t_1$ and that a careful investigation is necessary. For this purpose, it is convenient to introduce a scaled variable $z \equiv (t - t_1)/\sigma$.[17),18),19),20),21)] Then, one finds that $V_0(z,\sigma) \equiv v_0(t_1 + \sigma z, \sigma) - v_{00}(t_1 + \sigma z)$ and $U_0(z,\sigma) \equiv u_0(t_1 + \sigma z, \sigma) - u_{00}(t_1 + \sigma z)$ can be expanded into power series with respect to σ:

$$\begin{pmatrix} V_0(z,\sigma) \\ U_0(z,\sigma) \end{pmatrix} = \sum_{l=0}^{\infty} \sigma^l \begin{pmatrix} V_{0l}(z) \\ U_{0l}(z) \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} V_{0l}(z) \\ U_{0l}(z) \end{pmatrix} = \sum_{i=1}^{\infty} \frac{1}{z^{i-l}} \begin{pmatrix} a_l^{(i)} \\ b_l^{(i)} \end{pmatrix} \tag{3·5}$$

and that (V_{00}, U_{00}), (V_{01}, U_{01}), $\ldots$ correspond, respectively, to the most divergent term, the next divergent term $\ldots$ at $t = t_1$. It turns out that the expansions (3·5) of V_{0l}, U_{0l} are Borel summable asymptotic expansions which are valid in the sector Re $z < 0$. Hence, when one crosses the line Re $z = 0$, new terms would be added. These aspects will be discussed in the next section.

§4. Inner Equation and Analytic Continuation of Solutions

4.1. *Inner equation*

In terms of the variables $V(z,\sigma) \equiv v(t_1 + \sigma z, \sigma) - v_{00}(t_1 + \sigma z)$ and $U(z,\sigma) \equiv u(t_1 + \sigma z, \sigma) - u_{00}(t_1 + \sigma z)$, the Harper map (1·1) reads as

$$\begin{aligned} \Delta V(z,\sigma) &= -\sigma \sin\left(U(z,\sigma) + u_{00}(t_1 + \sigma z)\right) - \Delta v_{00}(t_1 + \sigma z) \\ \Delta U(z-1,\sigma) &= k\sigma \sin\left(V(z,\sigma) + v_{00}(t_1 + \sigma z)\right) - \Delta u_{00}(t_1 + \sigma(z-1)) \end{aligned} \tag{4·1}$$

where Δ stands for the difference operator: $\Delta f(z) \equiv f(z+1) - f(z)$. This equation is used to study the asymptotic property of the solutions for $|z| \to \infty$ *) and will be referred to as an inner equation. On the other hand, the map (1·1) in the original time variable is referred to as an outer equation.

Let $V_0(z,\sigma)$, $U_0(z,\sigma)$ be the solution of (4·1) corresponding to the perturbative solution and let

$$\begin{pmatrix} V_0(z,\sigma) \\ U_0(z,\sigma) \end{pmatrix} = \sum_{l=0}^{\infty} \sigma^l \begin{pmatrix} V_{0l}(z) \\ U_{0l}(z) \end{pmatrix} \tag{4·2}$$

be their expansions with respect to σ, then the equations for V_{0l}, U_{0l} are obtained by substituting (4·2) into (4·1). The first two sets are

$$\begin{aligned} \Delta V_{00}(z) &= i\frac{e^{-iU_{00}(z)}}{z} - i\ln\left(1 + \frac{1}{z}\right) \\ \Delta U_{00}(z) &= -i\frac{e^{iV_{00}(z+1)}}{z+1} + i\ln\left(1 + \frac{1}{z}\right) \end{aligned} \tag{4·3}$$

$$\begin{aligned} \Delta V_{01}(z) &= \left[i\frac{k-1}{2} + \frac{U_{01}(z)}{z}\right] e^{-iU_{00}(z)} - \frac{i}{2}(k-1) \\ \Delta U_{01}(z) &= \left[\frac{V_{01}(z+1)}{z+1} - i\frac{1-k}{2}\right] e^{iV_{00}(z+1)} + \frac{i}{2}(1-k) \end{aligned} \tag{4·4}$$

*) The asymptotic behavior for $|z| \to \infty$ corresponds to that for $t \to t_1$ in the original varibale.

From these equations, one can directly obtain the expansion (3·5) discussed in the previous section. Indeed, by posing the matching condition at $z = 0$:

$$\mathrm{Res}\begin{pmatrix} U_{0k}(z) \\ V_{0k}(z) \end{pmatrix}\Bigg|_{z=0} = \mathrm{Res}\begin{pmatrix} v_{0\,k+1}(t) \\ u_{0\,k+1}(t) \end{pmatrix}\Bigg|_{t=t_1}, \tag{4·5}$$

one has

$$\begin{aligned} V_{00}(z) &= \frac{i}{12z^2} - \frac{107i}{4320z^4} + \mathrm{O}\left(\frac{1}{z^5}\right) \\ U_{00}(z) &= -\frac{i}{2z} + \frac{i}{24z^2} + \frac{i}{24z^3} - \frac{191i}{8640z^4} + \mathrm{O}\left(\frac{1}{z^5}\right) \\ V_{01}(z) &= -\frac{i}{24}\frac{kt_1+1}{z} + \mathrm{O}\left(\frac{1}{z^3}\right) \\ U_{01}(z) &= \frac{i(k-1)}{4} + \frac{i}{24}\frac{k(t_1+1)}{z} - \frac{i}{48}\frac{k(t_1+1)}{z^2} + \mathrm{O}\left(\frac{1}{z^3}\right) \end{aligned}$$

In the following subsections, we shall show that these expansions are Borel summable and that their analytic continuations from the sector Re $z < 0$, Im $z < 0$ to Re $z > 0$, Im $z < 0$ give necessary information for the construction of the approximate unstable manifold for $0 <$ Re $t < 2T$. Note that , one can prove from the above asymptotic expansions and (4·3) that all the coefficient of $1/z$ exponents of $V_{00}(z), U_{00}(z)$ are purely imaginary.

4.2. *Borel transformation and analytic continuation*

We define their Borel transforms $\widetilde{V}_0(p)$, $\widetilde{U}_0(p)$ by

$$\begin{aligned} V_{00}(z) &\equiv L[\widetilde{V}_0(p)](z) \equiv \int_0^{-\infty} dp\, e^{-pz}\, \widetilde{V}_0(p) \\ U_{00}(z) &\equiv L[\widetilde{U}_0(p)](z) \equiv \int_0^{-\infty} dp\, e^{-pz}\, \widetilde{U}_0(p) \end{aligned} \tag{4·6}$$

The integration path is so chosen that the left hand sides are regular for Re $z \to -\infty$ since we are investigating the unstable manifold. As easily seen from (4·3), $\widetilde{V}_0(p)$, $\widetilde{U}_0(p)$ satisfy

$$\begin{aligned} -i(e^{-p}-1)\widetilde{V}_0(p) &= 1 + \int_0^p dp \left\{ \sum_{n=1}^{\infty} \frac{(-i)^n \widetilde{U}_0^{(*n)}(p)}{n!} \right\} - \frac{1-e^{-p}}{p} \\ i(1-e^{p})\widetilde{U}_0(p) &= 1 + \int_0^p dp \left\{ \sum_{n=1}^{\infty} \frac{i^n \widetilde{V}_0^{(*n)}(p)}{n!} \right\} - \frac{e^{p}-1}{p} \end{aligned} \tag{4·7}$$

where $\widetilde{V}_0^{(*n)}$ denotes the nth convolution recursively defined by

$$\widetilde{V}_0^{(*n)}(p) = \int_0^p dx \widetilde{V}_0(p-x)\widetilde{V}_0^{(*(n-1))}(x) \tag{4·8}$$

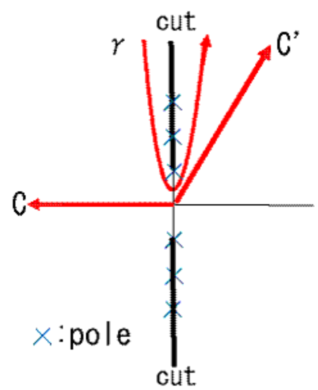

Fig. 3. Integration path in p-plane.

The factors $(e^{-p} - 1)$ and $(1 - e^{p})$ of (4·7) indicate that the Borel transforms $\widetilde{V}_0(p)$, $\widetilde{U}_0(p)$ may have singularities at $p = \pm 2\pi n i$ $(n = 1, 2, \cdots)$ and branch cuts starting from them (cf. Fig. 3).

Then, the analytic continuations of $V_{00}(z)$ and $U_{00}(z)$ from Re $z < 0$, Im $z < 0$ to Re $z > 0$, Im $z < 0$ are obtained by rotating the integration path in the p-plane from C to C′ as shown in Fig. 3. Note that $V_{00}(z)$ and $U_{00}(z)$ should be analytically continued along a curve passing below the singularity $z = 0$ where they have no cut (cf. Fig. 3). This corresponds to counterclockwise rotation of z, e.g., along a semicircle below $z = 0$ and, thus, one should rotate the p-integration path in a clockwise way (cf. Fig. 3). In short, the analytic continuation is given by

$$V_{00}(z) = \int_0^{-\infty} dp\, e^{-pz}\, \widetilde{V}_0(p) \to \int_0^{-\infty e^{i\theta}} dp\, e^{-pz}\, \widetilde{V}_0(p) - \int_\gamma dp\, e^{-pz}\, \widetilde{V}_0(p) \quad (4\cdot9)$$

where $\theta = \pi/2 + \epsilon'$ with ϵ' being a small positive number. The first term has the same asymptotic expansion with respect to z as in the domain Re $z < 0$, Im $z < 0$ and the second term is the desired additional term.

4.3. *Linearized equation*

If one can explicitly write down the Borel-transformed solution of the inner equation, one has only to change the integration path. But it is difficult to find the location and order of singularities in a straightforward way. Instead, we investigate them with the aid of the linearized inner equation.

As seen from the Borel transformed inner equation (4·7), singularities of the Borel transforms of $V_{00}(z)$ and $U_{00}(z)$ nearest to the real axis are located at $p = \pm 2\pi i$. According to the resurgence theory,[3),14)] the order of these singularities is given by the solution of the linearized equation of (4·3):

$$\Delta\Phi(z) = \frac{e^{-iU_{00}(z)}}{z}\Psi(z)\ , \qquad \Delta\Psi(z) = \frac{e^{iV_{00}(z+1)}}{z+1}\Phi(z+1)\ . \quad (4\cdot10)$$

More precisely, let $(V_A(z), U_A(z))$ and $(V_B(z), U_B(z))$ be two linearly independent

solutions of (4·10) and let $(\widetilde{V}_A(p), \hat{U}_A(p))$ and $(\hat{V}_B(p), \hat{U}_B(p))$ be the functions satisfying

$$e^{-2\pi iz}\begin{pmatrix} V_\alpha(z) \\ U_\alpha(z) \end{pmatrix} = -\int_\gamma dpe^{-pz}\begin{pmatrix} \hat{V}_\alpha(p) \\ \hat{U}_\alpha(p) \end{pmatrix} \qquad (\alpha = A, B), \tag{4·11}$$

then, with appropriately chosen constants Λ_0^A and Λ_0^B, the Borel transforms of $V_{00}(z)$ and $U_{00}(z)$ are given by

$$\begin{pmatrix} \widetilde{V}_0(p) \\ \widetilde{U}_0(p) \end{pmatrix} = \Lambda_0^A \begin{pmatrix} \hat{V}_A(p) \\ \hat{U}_A(p) \end{pmatrix} + \Lambda_0^B \begin{pmatrix} \hat{V}_B(p) \\ \hat{U}_B(p) \end{pmatrix} + \begin{pmatrix} R_v(p) \\ R_u(p) \end{pmatrix} \tag{4·12}$$

in a neighborhood of a singular point $p = 2\pi i$, where the functions $R_v(p)$, $R_u(p)$ are regular near $p = 2\pi i$.

With the aid of this observation, the Borel transforms $\widetilde{V}_0(p)$ and $\widetilde{U}_0(p)$ are determined as follows: At first, with the aid of the formulas*)

$$\begin{aligned} -\int_\gamma dp\, e^{-pz}\left[\frac{(j-1)!}{(p-2\pi ni)^j}\right] &= -2\pi i(-z)^{j-1}e^{-2\pi inz} \\ -\int_\gamma dp\, e^{-pz}\left[\frac{(p-2\pi in)^j \ln(p-2\pi in)}{j!}\right] &= \frac{-2\pi i}{z^{j+1}}e^{-2\pi inz} \end{aligned} \tag{4·13}$$

and the relation (4·11), one derives approximate functional forms of $\hat{V}_\alpha(p)$, $\hat{U}_\alpha(p)$ $(\alpha = A, B)$ near $p = 2\pi i$ from the asymptotic $1/z$-expansion of the solution of (4·10).

As pointed out at the end of Sec. 4.1, all the coefficients of $1/z$-expansions of $V_{00}(z)$ and $U_{00}(z)$ are purely imaginary and this implies that $\widetilde{V}_0(p)$, $\widetilde{U}_0(p)$ are purely imaginary for real p. By taking this property into account, the functional forms of $\widetilde{V}_0(p)$, $\widetilde{U}_0(p)$ are guessed from (4·12). Next, the power series solution of the Borel transformed inner equation (4·7) is obtained numerically. Finally, the coefficients Λ_0^A and Λ_0^B are determined by comparing the power series solution of $\widetilde{V}_0(p)$, $\widetilde{U}_0(p)$ and the power series expansion of their guessed functional forms. This calculation will be carried out in the next subsection.

In the rest of this subsection, we give a plausible argument why the Borel transformed inner solution near the singularity can be determined from the linearized equation. We restrict ourselves to the case where the Borel transformed solution has the form**):

$$\begin{pmatrix} \widetilde{V}_0(p) \\ \widetilde{U}_0(p) \end{pmatrix} = \sum_{n=1}^{\infty}\begin{pmatrix} g_n^v(p-2\pi ni) + h_n^v(p-2\pi ni)\ln(p-2\pi ni) \\ g_n^u(p-2\pi ni) + h_n^u(p-2\pi ni)\ln(p-2\pi ni) \end{pmatrix} \tag{4·14}$$

where $g_n^\alpha(p)$ $(\alpha = u, v)$ are meromorphic with a pole only at the origin and $h_n^\alpha(p)$ $(\alpha = u, v)$ are analytic and $e^{-pz}h_n^\alpha(p)(\mathrm{Im}\ z < 0)$ are rapidly decreasing for $p \to i\infty$.

*) We have chosen the branch cut of the logarithm along the positive imaginary axis as shown in Fig. 3. Note that this choice fixes the Stokes line on the negative imaginary axis in the z-plane, which corresponds to the line in the original t-plane joining t_1 and its complex conjugate.

**) For the Hénon map, Gelfreich and Sauzin[14] have shown that the solution of the Borel transformed inner equation has such a form in a neighborhood of the singularity.

Then, their contour integrals along the path γ are given by

$$-\int_\gamma dp e^{-pz}\begin{pmatrix}\widetilde{V}_0(p)\\ \widetilde{U}_0(p)\end{pmatrix}=\sum_{n=1}^{\infty}e^{-2\pi inz}\begin{pmatrix}V_{n0}(z)\\ U_{n0}(z)\end{pmatrix} \tag{4·15}$$

where

$$\begin{pmatrix}V_{n0}(z)\\ U_{n0}(z)\end{pmatrix}=\frac{2\pi}{i}\left[e^{2\pi inz}\mathrm{Res}\left[e^{-pz}\begin{pmatrix}g_n^v(p)\\ g_n^u(p)\end{pmatrix}\right]_{p=0}+\int_0^\infty dq e^{-iqz}\begin{pmatrix}h_n^v(iq)\\ h_n^u(iq)\end{pmatrix}\right] \tag{4·16}$$

The first terms of the right hand side of (4·16) are polynomials of z and the second terms can be expanded into a power series with respect to $1/z$. On the other hand, since

$$\begin{pmatrix}V_{00}(z)-\int_\gamma dp e^{-pz}\widetilde{V}_0(p)\\ U_{00}(z)-\int_\gamma dp e^{-pz}\widetilde{U}_0(p)\end{pmatrix}=\begin{pmatrix}V_{00}(z)\\ U_{00}(z)\end{pmatrix}+\sum_{n=1}^{\infty}e^{-2\pi inz}\begin{pmatrix}V_{n0}(z)\\ U_{n0}(z)\end{pmatrix}$$

and $(V_{00}(z), U_{00}(z))$ are solutions of (4·3), by substituting the above expansion into (4·3) and comparing the terms with the same powers in $e^{-2\pi iz}$, one finds that $V_{10}(z)$, $U_{10}(z)$ satisfy the linearized equation (4·10) and, thus, are linear combinations of $(V_A(z), U_A(z))$ and $(V_B(z), U_B(z))$. As a consequence, there exist constants Λ_0^A and Λ_0^B such that

$$\begin{aligned}&-\int_\gamma dp e^{-pz}\begin{pmatrix}g_1^v(p-2\pi i)+h_1^v(p-2\pi i)\ln(p-2\pi i)\\ g_1^u(p-2\pi i)+h_1^u(p-2\pi i)\ln(p-2\pi i)\end{pmatrix}\\ &=e^{-2\pi iz}\begin{pmatrix}V_{10}(z)\\ U_{10}(z)\end{pmatrix}=e^{-2\pi iz}\left\{\Lambda_0^A\begin{pmatrix}V_A(z)\\ U_A(z)\end{pmatrix}+\Lambda_0^B\begin{pmatrix}V_B(z)\\ U_B(z)\end{pmatrix}\right\}.\end{aligned} \tag{4·17}$$

This relation corresponds to (4·12) and implies that the functional form of the Borel transformed solution $\widetilde{V}_0(p)$, $\widetilde{U}_0(p)$ can be determined by the linearized equation.

4.4. *Determination of* $\widetilde{V}_0(p)$, $\widetilde{U}_0(p)$

Now we calculate $\widetilde{V}_0(p)$, $\widetilde{U}_0(p)$ following the procedure described in the previous subsection. Firstly, the linearized equation (4·11) has two linearly independent solutions $(V_A(z), U_A(z))$ and $(V_B(z), U_B(z))$, which can be expanded for large z as

$$\begin{aligned}V_A(z)&=-\frac{1}{z}+\frac{1}{6z^3}+\cdots, & U_A(z)&=\frac{1}{z}-\frac{1}{2z^2}+\frac{1}{12z^3}+\cdots\\ V_B(z)&=z-\frac{1}{540z^3}+\cdots, & U_B(z)&=z+\frac{1}{2}+\frac{1}{12z}-\frac{1}{24z^2}+\frac{11}{2160z^3}+\cdots\end{aligned} \tag{4·18}$$

Then, (4·13) leads to

$$\begin{aligned}\hat{V}_A(p)&=\frac{1}{2\pi i}\ln(p-2\pi i)+\cdots\\ \hat{V}_B(p)&=\frac{1}{2\pi i}\frac{1}{(p-2\pi i)^2}+\cdots\end{aligned} \tag{4·19}$$

where the omitted terms correspond to z^{-m} $(m \geq 2)$. Hence, one has

$$\begin{aligned}\widetilde{V}_0(p) &= \Lambda_0^A \hat{V}_A(p) + \Lambda_0^B \hat{V}_B(p) + \cdots \\ &= \frac{1}{2\pi i}\left\{\frac{\Lambda_0^B}{(p-2\pi i)^2} + \Lambda_0^A \ln(p-2\pi i)\right\} + \cdots \end{aligned} \qquad (4\cdot 20)$$

where the rest part is either regular or less singular at $p = 2\pi i$. As mentioned at the end of Sec. 4.1, all the coefficients of the $1/z$-expansion of $V_{00}(z)$ are purely imaginary and, thus, $\widetilde{V}_0(p)$ is purely imaginary for real p. This property determines a part of the regular components of $\widetilde{V}_0(p)$ and, as a result, one can guess

$$\begin{aligned}\widetilde{V}_0(p) &= \frac{1}{2\pi i}\left\{\frac{\Lambda_0^B}{(p-2\pi i)^2} + \frac{\Lambda_0^{B*}}{(p+2\pi i)^2} + \Lambda_0^A \ln(p-2\pi i) + \Lambda_0^{A*}\ln(p+2\pi i)\right\} + \cdots \\ &= \mathrm{Re}\Lambda_0^B\ f_1^{(R)}(p) + \mathrm{Im}\Lambda_0^B\ f_1^{(I)}(p) - \mathrm{Re}\Lambda_0^A f_{-1}^{(R)}(p) - \mathrm{Im}\Lambda_0^A f_{-1}^{(I)}(p) + \cdots \end{aligned} \qquad (4\cdot 21)$$

where the following auxiliary functions are introduced

$$\begin{aligned} f_j^{(R)}(p) &= \frac{j!}{2\pi i}\left(\frac{1}{(p-2\pi i)^{j+1}} + \frac{1}{(p+2\pi i)^{j+1}}\right) \quad (j = 0,1,2,\cdots) \\ f_j^{(I)}(p) &= \frac{j!}{2\pi}\left(\frac{1}{(p-2\pi i)^{j+1}} - \frac{1}{(p+2\pi i)^{j+1}}\right) \quad (j = 0,1,2,\cdots) \\ f_{-1}^{(R)}(p) &= \frac{i}{2\pi}\left[\ln(p-2\pi i) + \ln(p+2\pi i)\right] \\ f_{-1}^{(I)}(p) &= -\frac{1}{2\pi}\left[\ln(p-2\pi i) - \ln(p+2\pi i)\right] \end{aligned}$$

We note that the above auxiliary functions have power series of p with purely imaginary coefficients. One can easily show that the neglected terms of (4·21) are either regular or less singular at $p = \pm 2\pi i$. As easily seen, (4·21) admits the power series expansion near $p = 0$:

$$\begin{aligned}\widetilde{V}_0(p) = M &+ \sum_{n=0}^{\infty}\left\{\frac{i\mathrm{Im}\Lambda_0^B}{4\pi^3}(2n+2) + \frac{i\mathrm{Im}\Lambda_0^A}{\pi(2n+1)}\right\}(-1)^n\left(\frac{p}{2\pi}\right)^{2n+1} \\ &+ \sum_{n=1}^{\infty}\left\{\frac{i\mathrm{Re}\Lambda_0^B}{4\pi^3}(2n+1) + \frac{i\mathrm{Re}\Lambda_0^A}{2\pi n}\right\}(-1)^n\left(\frac{p}{2\pi}\right)^{2n}, \end{aligned} \qquad (4\cdot 22)$$

where $M = -\frac{i\mathrm{Im}\Lambda_0^A}{4} + \frac{i\mathrm{Re}\Lambda_0^B}{4\pi^3} - \frac{i\mathrm{Re}\Lambda_0^A}{2\pi}\log(4\pi^2)$. In a similar way, one can guess

$$\begin{aligned}\widetilde{U}_0(p) = M' &+ \sum_{n=0}^{\infty}\left\{\frac{i\mathrm{Im}\Lambda_0^B}{4\pi^3}(2n+2) - \frac{i\mathrm{Im}\left(\Lambda_0^A + \Lambda_0^B/12\right)}{\pi(2n+1)} + \frac{i\mathrm{Re}\Lambda_0^B}{4\pi^2}\right\}(-1)^n\left(\frac{p}{2\pi}\right)^{2n+1} \\ &+ \sum_{n=1}^{\infty}\left\{\frac{i\mathrm{Re}\Lambda_0^B}{4\pi^3}(2n+1) - \frac{i\mathrm{Re}\left(\Lambda_0^A + \Lambda_0^B/12\right)}{2\pi n} - \frac{i\mathrm{Im}\Lambda_0^B}{4\pi^2}\right\}(-1)^n\left(\frac{p}{2\pi}\right)^{2n} \end{aligned} \qquad (4\cdot 23)$$

where $M' = M + \frac{i\mathrm{Im}\ (2\Lambda_0^A + \Lambda_0^B/12)}{4} - \frac{i\mathrm{Im}\Lambda_0^B}{4\pi^2}$.

Now we numerically solve the Borel transformed inner equation (4·7). By substituting the power series expansions:

$$\widetilde{V}_0(p) \equiv \sum_{n=0}^{\infty} a_n p^n, \quad \widetilde{U}_0(p) \equiv \sum_{n=0}^{\infty} b_n p^n; \quad a_0 = 0, \quad b_0 = -\frac{i}{2}$$

into (4·7) and comparing term by term, the coefficients $a_n,\ b_n$ are determined recursively and one has

$$\begin{aligned} i a_{2n+1}(-1)^n (2\pi)^{2n+1} &\to -(2n+2)A_1 + \frac{A_2}{2n+1} \quad (n \to \infty) \\ i a_{2n} &= 0, \quad (\forall n) \end{aligned} \tag{4·24}$$

$$\begin{aligned} i b_{2n+1}(-1)^n (2\pi)^{2n+1} &\to -(2n+2)A_1 + \frac{A_2}{2n+1} \quad (n \to \infty) \\ i b_{2n}(-1)^n (2\pi)^{2n} &\to A_3 \quad (n \to \infty) \end{aligned} \tag{4·25}$$

where $A_1 = 0.27893,\ A_2 = 0.417,\ A_3 = 0.87628.$

By comparing (4·22) with (4·24), and (4·23) with (4·25), one finds that Λ_0^A and Λ_0^B are purely imaginary and

$$\Lambda_0^B = 4\pi^3 i A_1\ , \quad \Lambda_0^A + \frac{\Lambda_0^B}{12} = -\Lambda_0^A = \pi i A_2\ , \quad \Lambda_0^B = 4\pi^2 i A_3\ . \tag{4·26}$$

Because of the first and third equalities of (4·26), the ratio A_3/A_1 has to be $\pi\Lambda_0^B/\Lambda_0^B = \pi$ and the evaluated values give an excellent agreement: $A_3/A_1 = 3.1416$. Also, the first and second equalities require $(6A_2)/(\pi^2 A_1)$ to be unity and the evaluated values are consistent:$(6A_2)/(\pi^2 A_1) = 0.909$*).

Here we remark about the contributions from a singularity at $p = ip_c$ $(p_c > 2\pi)$. If the singularity is of the same type as that at $p = 2\pi i$, the contributions to the coefficient of p^n are exponentially smaller by a factor of $\left(\frac{2\pi}{p_c}\right)^n$ than those from $p = 2\pi i$. Thus, such contributions do not affect the present analysis and, at the same time, are difficult to be evaluated.

In short, following the procedure discussed in the previous subsection, we have obtained the Borel transforms $\widetilde{V}_0(p),\ \widetilde{U}_0(p)$:

$$\begin{aligned} \widetilde{V}_0(p) &= -\frac{\Lambda_0^A}{i} f_{-1}^{(I)}(p) + \frac{\Lambda_0^B}{i} f_1^{(I)}(p) \\ \widetilde{U}_0(p) &= -\frac{\Lambda_0^A}{i} f_{-1}^{(I)}(p) + \frac{\Lambda_0^B}{i} \left(f_1^{(I)}(p) - \frac{1}{2} f_0^{(I)}(p) \right) \end{aligned} \tag{4·27}$$

*) The numerical evaluations of $A_1,\ A_3$ are quite robust. On the other hand, A_2 is sensitive to an error of A_1 since the coefficients of $1/(2n+1)$ are fitted after subtracting the leading terms of order n. However, we observe that the coefficients of $1/(2n+1)$ in $a_{2n+1}(-1)^n(2\pi)^{2n+1}$ and $b_{2n+1}(-1)^n(2\pi)^{2n+1}$ are always the same, or the second equation of (4·26) holds. And the value of A_2 are fitted so that the first and second equations are well satisfied.

4.5. *Borel transforms of* $V_{01}(z)$ *and* $U_{01}(z)$

Let $V_{01}^-(z),\ U_{01}^-(z)$ be the sum of negative powers of z in the asymptotic expansion $V_{01}(z),\ U_{01}(z)$, and let $\widetilde{V}_1(p), \widetilde{U}_1(p)$ be the Borel transforms of $V_{01}^-(z),\ U_{01}^-(z)$ in the sector including $z=-\infty$, then they provide the following additional terms in the other sector including $z=+\infty$

$$-\int_\gamma dp\, e^{-pz} \begin{pmatrix} \widetilde{V}_1(p) \\ \widetilde{U}_1(p) \end{pmatrix} = e^{-2\pi i z} \begin{pmatrix} V_{11}(z) \\ U_{11}(z) \end{pmatrix} + \cdots \tag{4·28}$$

where the neglected terms are proportional to a factor of $e^{-ip_c z}$ $(p_c > 2\pi)$.

Similarly to the construction of $\widetilde{V}_0(p),\ \widetilde{U}_0(p)$, one can get $\widetilde{V}_1(p),\ \widetilde{U}_1(p)$ from $V_{11}(z),\ U_{11}(z)$, which obey:

$$\begin{aligned} \varDelta V_{11}(z) &= \frac{U_{11}(z) + \frac{k-1}{2} z U_{10}(z)}{z} e^{-iU_{00}(z)} - i \frac{U_{10}(z) U_{01}(z)}{z} e^{-iU_{00}(z)} \\ \varDelta U_{11}(z-1) &= \frac{V_{11}(z) + \frac{1-k}{2} z V_{10}(z)}{z} e^{iV_{00}(z)} + i \frac{V_{10}(z) V_{01}(z)}{z} e^{iV_{00}(z)} \end{aligned} \tag{4·29}$$

where V_{10}, U_{10} are introduced in (4·15).

Because of the linearity of (4·29), V_{11} and U_{11} are given by

$$\begin{pmatrix} V_{11}(z) \\ U_{11}(z) \end{pmatrix} = \varLambda_0^A \begin{pmatrix} V_{11}^A(z) \\ U_{11}^A(z) \end{pmatrix} + \varLambda_0^B \begin{pmatrix} V_{11}^B(z) \\ U_{11}^B(z) \end{pmatrix} + \varLambda_1^A \begin{pmatrix} V_A(z) \\ U_A(z) \end{pmatrix} + \varLambda_1^B \begin{pmatrix} V_B(z) \\ U_B(z) \end{pmatrix} \tag{4·30}$$

where the constants $\varLambda_0^\alpha$ $(\alpha = A, B)$ are introduced in the previous subsection, $\varLambda_1^\alpha$ $(\alpha = A, B)$ are new constants, and $V_{11}^\alpha(z), U_{11}^\alpha(z)$ $(\alpha = A, B)$ are solutions of

$$\begin{aligned} \varDelta V_{11}^\alpha(z) &= \frac{U_{11}^\alpha(z) + \frac{k-1}{2} z U_\alpha(z)}{z} e^{-iU_{00}(z)} - i \frac{U_\alpha(z) U_{01}(z)}{z} e^{-iU_{00}(z)} \\ \varDelta U_{11}^\alpha(z-1) &= \frac{V_{11}^\alpha(z) + \frac{1-k}{2} z V_\alpha(z)}{z} e^{iV_{00}(z)} + i \frac{V_\alpha(z) V_{01}(z)}{z} e^{iV_{00}(z)} \end{aligned} \tag{4·31}$$

such that the asymptotic expansions of $V_{11}^\alpha(z)$ $(\alpha = A, B)$ have no terms proportional to z nor $1/z$. Then, asymptotic expansions of $V_{11}(z),\ U_{11}(z)$ are obtained by exactly the same procedure as in the previous subsection and, for large $|z|$, we have

$$\begin{pmatrix} V_{11}(z) \\ U_{11}(z) \end{pmatrix} \approx \begin{pmatrix} \varLambda_1^B z + \frac{k-1}{6} \varLambda_0^B z^2 \\ \varLambda_1^B z - \frac{k-1}{6} \varLambda_0^B (z^2 + z) \end{pmatrix} \tag{4·32}$$

where the terms of order z^{-m} $(m \geq 0)$ are omitted since the numerical estimation of their contributions to $\widetilde{V}_1(p), \widetilde{U}_1(p)$ is very difficult[*)].

Now we go back to the equation (4·4) of $V_{01}(z), U_{01}(z)$. Because it is linear, one gets (See appendix D for more details.)

$$\begin{pmatrix} V_{01}^-(z) \\ U_{01}^-(z) \end{pmatrix} = i \left\{ \frac{kt_1 + 1}{24} \begin{pmatrix} V_A(z) \\ U_A(z) \end{pmatrix} + \frac{k-1}{2} \begin{pmatrix} V_B(z) - z \\ U_B(z) - (z - \frac{1}{2}) \end{pmatrix} \right\}. \tag{4·33}$$

[*)] Such an estimation requires a separation of the terms of order $1/n$ from the coefficients a_n, b_n of the power series expansion of $\widetilde{V}_1, \widetilde{U}_1$. But, since their leading order terms are proportional to n^2, it is very difficult.

As one can see from the definition of $\begin{pmatrix} V_\alpha(z) \\ U_\alpha(z) \end{pmatrix}$, the Borel transform of it has real values for real p. On the other hand, (4·32) shows that $\widetilde{V}_1(p)$, $\widetilde{U}_1(p)$ have the first and second order poles at $p = 2\pi i$. These observations imply

$$\begin{pmatrix} \widetilde{V}_1(p) \\ \widetilde{U}_1(p) \end{pmatrix} = -4\pi^4(k-1)B_1 f_2^{(I)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \left(\frac{kt_1+1}{12}\pi^3 B_4 + (k-1)\pi^3 B_2 \right) f_1^{(R)}(p) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ -\pi^3(k-1)B_3 f_1^{(I)}(p) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4·34}$$

$$\begin{pmatrix} V_{11}(z) \\ U_{11}(z) \end{pmatrix} = 4\pi^4 i(k-1)B_1 z^2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ - \left(\frac{kt_1+1}{12}\pi^3 B_4 + (k-1)\pi^3 B_2 \right) z \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \pi^3 i(k-1)B_3 z \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4·35}$$

The constants B_1, B_2, B_3 and B_4 can be numerically evaluated by considering the Borel transformations of V_A, U_A, $V_B - z$ and $U_B - z - 1/2$ (for more detail, see Appendix D). Then, we have

$$B_1 = 0.01480, \quad B_2 = 0.14, \; B_3 = 0.186, \; B_4 = 3.503 \tag{4·36}$$

By substituting (4·35) into (4·32), one gets

$$\begin{aligned} \Lambda_0^B &= 24\pi^4 i B_1 = 6\pi^3 i B_3 \\ \Lambda_1^B &= -(k-1)\pi^3 B_2 - \frac{kt_1+1}{12}\pi^3 B_4. \end{aligned} \tag{4·37}$$

Eq. (4·37) and $\Lambda_0^B = i4\pi^3 A_1$ imply two relations among A_1, B_2 and B_4, which are satisfied by the present numerical estimations rather well:

$$\frac{A_1}{6\pi B_1} = 0.99987 \simeq 1 \qquad \frac{2A_1}{3\pi B_3} = 0.99974 \simeq 1 \; .$$

Hence, we obtain

$$\begin{aligned} \widetilde{V}_1(p) &= i\frac{\Lambda_0^B(k-1)}{6} f_2^{(I)}(p) + \Lambda_1^B f_1^{(R)}(p) \\ \widetilde{U}_1(p) &= -i\frac{\Lambda_0^B(k-1)}{6} \left(f_2^{(I)}(p) - f_1^{(I)}(p) \right) + \Lambda_1^B f_1^{(R)}(p) \end{aligned} \tag{4·38}$$

4.6. *Asymptotic expansions of the additional terms*

In summary, we have shown, up to the order of σ^1, that $V(z,\sigma), U(z,\sigma)$ acquire new terms in the sector including $z = +\infty$

$$\begin{pmatrix} V(z,\sigma) \\ U(z,\sigma) \end{pmatrix} = \begin{pmatrix} V_{00}(z) + \sigma V_{01}(z) \\ U_{00}(z) + \sigma U_{01}(z) \end{pmatrix} + e^{-2\pi i z} \begin{pmatrix} V_{10}(z) + \sigma V_{11}(z) \\ U_{10}(z) + \sigma U_{11}(z) \end{pmatrix} + \cdots \tag{4·39}$$

where the first term corresponds to the perturbative solution studied in the previous section and the second term is the additional term:

$$V_{10}(z) + \sigma V_{11}(z) \approx \left(-\frac{\Lambda_0^A}{z} + \Lambda_0^B z\right) + \sigma\left(\Lambda_1^B z + \Lambda_0^B \frac{k-1}{6} z^2\right) \tag{4·40}$$

$$U_{10}(z) + \sigma U_{11}(z) \approx \left[\frac{\Lambda_0^A}{z} + \Lambda_0^B\left(z + \frac{1}{2} + \frac{1}{12z}\right)\right] + \sigma\left[\Lambda_1^B z - \Lambda_0^B \frac{k-1}{6}(z^2 + z)\right] \tag{4·41}$$

Although the functional forms of $V_{10}, U_{10}, V_{11}, U_{11}$ can be derived from their linear equations, the resurgence theory and Borel resummation play an essential role for the determination of the coefficients $\Lambda_0^A, \Lambda_0^B, \Lambda_1^B$.

In the next section, only the terms of order of z and z^2 are used to derive the solutions of the outer equation in the domain Re $t > 0$ since σ/z-terms are not determined because of the nemerical difficulty. However, it should be noted that the existence of $1/z$ terms does show the fact that the point $p = 2\pi i$ is a branch point of the Borel transforms of $V(z,\sigma), U(z,\sigma)$. Before closing this section, we remark that (4·40) can be rewritten as

$$V_{10}(z) + \sigma V_{11}(z) \approx -\frac{\Lambda_0^A}{z} + (\Lambda_0^B + \sigma\Lambda_1^B)\left(z + \sigma\frac{k-1}{6}z^2\right).$$

§5. Matching between Inner and Outer Solutions

5.1. *Matching at the singular point $t = t_1$*

In this section, solutions of the outer equations are constructed and they are matched with the inner solutions. We first consider the contribution from the singularity $t = t_1$. Corresponding to the expansion of the analytically continued inner solutions: $V = V_0 + e^{-2\pi i z}V_1 + \cdots$, $U = U_0 + e^{-2\pi i z}U_1 + \cdots$, the original solutions v and u acquire new terms in a sector Re $t \geq$ Re $t_1 = 0$ of the t_1-neighborhood:

$$\begin{aligned} v(t) &= v_0(t,\sigma) + v_1(t,\sigma)e^{-\frac{2\pi i}{\sigma}t} + \cdots \\ u(t) &= u_0(t,\sigma) + u_1(t,\sigma)e^{-\frac{2\pi i}{\sigma}t} + \cdots \end{aligned} \tag{5·1}$$

Because of $e^{-2\pi i z} = e^{-\frac{\pi^2}{\sigma\sqrt{k}}}e^{-\frac{2\pi i}{\sigma}t}$, v_1 and u_1 are exponentially small with respect to σ. By substituting (5·1) into (1·1) and comparing term by term, we obtain the equations for v_1 and u_1:

$$\begin{aligned} v_1(t+\sigma) - v_1(t) &= -\sigma u_1(t)\cos u_0(t) \\ u_1(t+\sigma) - u_1(t) &= k\sigma v_1(t+\sigma)\cos v_0(t+\sigma) \end{aligned} \tag{5·2}$$

Its solution is uniquely determined by the matching condition:

$$e^{-\frac{2\pi i t}{\sigma}}\begin{pmatrix} v_1(t,\sigma) \\ u_1(t,\sigma)\end{pmatrix}\Bigg|_{t=t_1+\sigma z} = e^{-2\pi i z}\begin{pmatrix} V_{10}(z) + \sigma V_{11}(z) + \cdots \\ U_{10}(z) + \sigma U_{11}(z) + \cdots \end{pmatrix} \tag{5·3}$$

or equivalently

$$\begin{pmatrix} v_1(t_1+\sigma z,\sigma) \\ u_1(t_1+\sigma z,\sigma) \end{pmatrix} = e^{\frac{2\pi i t_1}{\sigma}} \begin{pmatrix} V_{10}(z)+\sigma V_{11}(z)+\cdots \\ U_{10}(z)+\sigma U_{11}(z)+\cdots \end{pmatrix} \tag{5·4}$$

This condition suggests the following expansions:

$$v_1(t,\sigma) = \sigma^j e^{\frac{2\pi i t_1}{\sigma}} \sum_{n=0}^{\infty} v_{1n}(t)\sigma^n \ , \qquad u_1(t,\sigma) = \sigma^j e^{\frac{2\pi i t_1}{\sigma}} \sum_{n=0}^{\infty} u_{1n}(t)\sigma^n \tag{5·5}$$

where the exponent j will be determined later. Then, equations for $v_{10}, u_{10}, v_{11}, u_{11}$ are given by

$$\begin{aligned} &v_{10}'(t) = -u_{10}(t)\cos u_{00}(t) \\ &u_{10}'(t) = k v_{10}(t)\cos v_{00}(t) \end{aligned} \tag{5·6}$$

$$\begin{aligned} &v_{11}'(t) + \frac{1}{2}v_{10}''(t) = -u_{11}(t)\cos u_{00}(t) + u_{10}(t)u_{01}(t)\sin u_{00}(t) \\ &u_{11}'(t) - \frac{1}{2}u_{10}''(t) = k\left(v_{11}(t)\cos v_{00}(t) - v_{10}(t)v_{01}(t)\sin v_{00}(t)\right) \end{aligned} \tag{5·7}$$

First, we solve (5·6). Since it is linear, one has

$$\begin{pmatrix} v_{10}(t) \\ u_{10}(t) \end{pmatrix} = a \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} + b \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} \tag{5·8}$$

where $x_1(t), y_1(t), x_2(t), y_2(t)$ are defined by (2·14) and (2·15).

On the other hand, up to the order of σ^0, the matching condition leads to

$$\begin{aligned} \sigma^j \left.\begin{pmatrix} v_{10}(\sigma z+t_1) \\ u_{10}(\sigma z+t_1) \end{pmatrix}\right|_{\sigma^0} &= \sigma^j \left.\left\{ a \begin{pmatrix} x_1(\sigma z+t_1) \\ y_1(\sigma z+t_1) \end{pmatrix} + b \begin{pmatrix} x_2(\sigma z+t_1) \\ y_2(\sigma z+t_1) \end{pmatrix} \right\}\right|_{\sigma^0} \\ &\approx \left.\begin{pmatrix} V_{10}(z)+\sigma V_{11}(z)+\cdots \\ U_{10}(z)+\sigma U_{11}(z)+\cdots \end{pmatrix}\right|_{\text{dom.}} = \Lambda_0^B \begin{pmatrix} z \\ z \end{pmatrix} \end{aligned} \tag{5·9}$$

where the subscript σ^0 indicates to take the terms of order σ^0 and 'dom.' stands for the largest part for $z \to \infty$. Because the left hand side of (5·9) starts from z, one should have $j = -1$. And the coefficients a and b are determined by the requirement that (5·9) admits well defined $\sigma \to 0$ limit:

$$a = 0 \ , \qquad b = \frac{2\Lambda_0^B}{i} \ .$$

Therefore, up to the 0th order in σ, we have

$$\begin{pmatrix} v_1(t) \\ u_1(t) \end{pmatrix} \approx \frac{2\Lambda_0^B}{i\sigma} e^{-\frac{2\pi i(t-t_1)}{\sigma}} \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} \tag{5·10}$$

Similarly, up to the first order in σ, one obtains

$$\begin{pmatrix} v_1(t) \\ u_1(t) \end{pmatrix} \approx \frac{2\Lambda^{(1)}}{i\sigma} e^{-\frac{2\pi i(t-t_1)}{\sigma}} \begin{pmatrix} x_2(t) \\ y_2(t)+\sigma y_2'(t)/2 \end{pmatrix} \tag{5·11}$$

where $\Lambda^{(1)} \equiv \Lambda_0^B + \sigma\Lambda_1^B$. Matching of higher order terms with respect to σ is discussed in Appendix F.

5.2. *Contribution from the singular point at $t = t_1^*$*

So far, the behavior of the solution near $t = t_1$ has been considered. In order to derive the solution valid in the domain Re $t > 0$, the contribution from the complex conjugate singularity $t = t_1^*$ should be taken into account. Because $v_0(t), u_0(t)$ is real analytic, it is simply the complex conjugate of the contribution from t_1 (See appendix E). Hence, the unstable manifold for $t < 2T$ is described by

$$\begin{pmatrix} v_u(t) \\ u_u(t) \end{pmatrix} = \begin{pmatrix} v_{00}(t) + \sigma^2 v_{02}(t) \\ u_{00}(t) + \sigma \frac{y_1(t)}{2} + \sigma^2 u_{02}(t) + \sigma^3 \left(\frac{1}{2} u_{02}'(t) - \frac{1}{24} y_1''(t) \right) \end{pmatrix} \\ + S(t) \mathrm{Re} \left[\frac{4\Lambda^{(1)}}{i\sigma} e^{-\frac{2\pi i (t-t_1)}{\sigma}} \begin{pmatrix} x_2(t) \\ y_2(t) + \sigma y_2'(t)/2 \end{pmatrix} \right] \qquad (5\cdot12)$$

where $S(t)$ denotes a step function.

§6. Contributions from Other Singularities

So far, contributions from t_1 and t_1^* have been considered. Here, we study contributions from other singular points and higher order terms. As easily seen, in the domain Re $t > 0$, the expression (5·12) of the unstable manifold has a singularity at

$$t \equiv t_2 = 2T + \frac{1}{\sqrt{k}} \left(n + \frac{1}{2} \right) \pi i \ .$$

Therefore, the solution would change its form when t exceeds $2T$. As in the previous analysis, the additional terms can be found from the asymptotic behavior of the solution near t_2 and its complex conjugate t_2^*.

The solution (5·12) valid in $t < 2T$ is a sum of the perturbative solution and the terms proportional to $S(t)$, and each term produces a new additional term in the sector Re $t > 2T$. We analyze these two terms separately.

(1) The additional terms arising from the perturbative solution.

The additional terms arising from the perturbative solution can be obtained exactly in the same way as those from $t = t_1$ and we have

$$\begin{pmatrix} V_{01}^-(z) \\ U_{01}^-(z) \end{pmatrix} = i \left\{ \frac{kt_2 - 1}{24} \begin{pmatrix} V_A(z) \\ -U_A(z) \end{pmatrix} + \frac{k-1}{2} \begin{pmatrix} -(V_B(z) - z) \\ U_B(z) - (z - \frac{1}{2}) \end{pmatrix} \right\} \qquad (6\cdot1)$$

$$\begin{pmatrix} v_1^{(2)}(t) \\ u_1^{(2)}(t) \end{pmatrix} = \mathrm{Re} \left[\frac{4\Lambda^{(2)}}{i\sigma} e^{-\frac{2\pi i (t-t_2)}{\sigma}} \left\{ \frac{T(k-1)^2 - (1+k)}{4k(k-1)} \begin{pmatrix} x_1(t) \\ y_1(t) + \sigma y_1'(t)/2 \end{pmatrix} \right. \right. \\ \left. \left. + \begin{pmatrix} x_2(t) \\ y_2(t) + \sigma y_2'(t)/2 \end{pmatrix} \right\} \right], \qquad (6\cdot2)$$

where $\Lambda^{(2)}$ is given by $\Lambda^{(2)} \equiv -i4\pi^3 A_1 + \sigma\pi^3 \left\{ -(k-1)B_2 + \frac{kt_2 - 1}{12} B_4 \right\}$.

(2) The additional terms arising from the term proportional to $S(t)$.

Let us prove that the terms added in the domain $0 <$ Re$t < 2T$ generate

new terms which are of order of ϵ^2 and, thus, are negligible. With the same analysis as for the most divergent terms from the perturbative solution, their most divergent terms near $t = t_2$ are found to be

$$\frac{i\Gamma}{\sigma}\xi(t)\begin{pmatrix} V_A(z') \\ -U_A(z') \end{pmatrix} - \frac{i\Gamma\sigma}{2}\cos\frac{2\pi t}{\sigma}\begin{pmatrix} V_B(z') \\ -U_B(z') \end{pmatrix}, \tag{6·3}$$

where $\xi(t), \Gamma$ are defined by

$$\xi(t) \equiv -\frac{t_2(1-k)^2 - 2(1+k)}{8k(k-1)}\cos\frac{2\pi t}{\sigma} + \frac{t_1(k-1)}{8ik}\sin\frac{2\pi t}{\sigma}$$
$$\Gamma \equiv \frac{4\Lambda_0^B}{i\sigma}e^{-\frac{\pi^2}{\sqrt{k}\sigma}} \tag{6·4}$$

This can be represented with the aid of the Borel transformations:

$$\frac{i\Gamma}{\sigma}\xi(t)\int_0^{-\infty} dp\, e^{-pz'}\begin{pmatrix} \widetilde{V}_A(p) \\ -\widetilde{U}_A(p) \end{pmatrix} - \frac{i\Gamma\sigma}{2}\cos\frac{2\pi t}{\sigma}\left[\int_0^{-\infty} dp\, e^{-pz'}\begin{pmatrix} \widetilde{V}_B(p) \\ -\widetilde{U}_B(p) \end{pmatrix} + \begin{pmatrix} z' \\ z'+1/2 \end{pmatrix}\right]$$

where $(\widetilde{V}_A(p), \widetilde{U}_A(p))$ and $(\widetilde{V}_B(p), \widetilde{U}_B(p))$ are, respectively, the Borel transforms of $(V_A(z'), U_A(z'))$ and $(V_B(z') - z', U_B(z') - z' - 1/2)$. Thus the following term is added in $\mathrm{Re}t > 2T$

$$\begin{aligned}
&-\frac{i\Gamma}{\sigma}\xi(t)\int_\gamma dp\, e^{-pz'}\begin{pmatrix} \widetilde{V}_A(p) \\ -\widetilde{U}_A(p) \end{pmatrix} + \frac{i\Gamma\sigma}{2}\cos\frac{2\pi t}{\sigma}\int_\gamma dp\, e^{-pz'}\begin{pmatrix} \widetilde{V}_B(p) \\ -\widetilde{U}_B(p) \end{pmatrix} \\
=&\Gamma e^{-2\pi i z'}\left[\frac{2\pi^3 B_4 z'}{\sigma}\xi(t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right. \\
&\left.+\frac{i\sigma}{2}\cos\frac{2\pi t}{\sigma}\left\{8\pi^4 B_1 z'^2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\pi^3 i B_2 z'\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 2\pi^3 B_3 z'\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}\right] \\
\simeq&\Gamma e^{-2\pi i z'}\left[\frac{2\pi^3 B_4(t-t_2)}{\sigma^2}\xi(t)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{i}{2\sigma}(t-t_2)^2\cos\frac{2\pi t}{\sigma}8\pi^4 B_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right],
\end{aligned}$$

which leads to terms of order of $\frac{1}{\sigma^3}e^{-\frac{2\pi^2}{\sqrt{k}\sigma}}$ in the outer solution. Therefore this term is negligible when $\dfrac{1}{\sigma^3}e^{-\frac{2\pi^2}{\sqrt{k}\sigma}} \ll \dfrac{1}{\sigma}e^{-\frac{\pi^2}{\sqrt{k}\sigma}}$.

As a result, the overall contribution of the singular points is the simple sum of the contributions from $t_1,\ t_2,\ t_1^*,\ t_2^*$. Because (6·2) has no singularity in the domain $\mathrm{Re}\, t > 2T$, one finally has the following expression of the unstable manifold valid in the whole time domain:

$$\begin{aligned}
v_u(t) =& v_{00}(t) + \sigma^2 v_{02}(t) \\
&+ S(t)\mathrm{Re}\left[\frac{4\Lambda^{(1)}}{i\sigma}e^{\frac{2\pi i t_1}{\sigma}}x_2(t)e^{-\frac{2\pi i t}{\sigma}}\right]
\end{aligned}$$

$$
\begin{aligned}
&\quad + S_+(t)\mathrm{Re}\left[\frac{4\Lambda^{(2)}}{i\sigma}e^{\frac{2\pi i t_2}{\sigma}}\left\{\left(\frac{T(k-1)^2-(1+k)}{4k(k-1)}\right)x_1(t)+x_2(t)\right\}e^{-\frac{2\pi i t}{\sigma}}\right] \\
u_u(t) &= u_{00}(t)+\sigma\frac{y_1(t)}{2}+\sigma^2 u_{02}(t)+\sigma^3\left(\frac{1}{2}u_{02}'(t)-\frac{1}{24}y_1''(t)\right) \\
&\quad + S(t)\mathrm{Re}\left[\frac{4\Lambda^{(1)}}{i\sigma}e^{\frac{2\pi i t_1}{\sigma}}\left(y_2(t)+\sigma\frac{y_2'(t)}{2}\right)e^{-\frac{2\pi i t}{\sigma}}\right] \\
&\quad + S_+(t)\mathrm{Re}\left[\frac{4\Lambda^{(2)}}{i\sigma}e^{\frac{2\pi i t_2}{\sigma}}\left\{\left(\frac{T(k-1)^2-(1+k)}{4k(k-1)}\right)\left(y_1(t)+\sigma\frac{y_1'(t)}{2}\right)\right.\right. \\
&\qquad\qquad \left.\left. +\left(y_2(t)+\sigma\frac{y_2'(t)}{2}\right)\right\}e^{-\frac{2\pi i t}{\sigma}}\right] \qquad (6\cdot5)
\end{aligned}
$$

where $S_+(t) = S(t-2T)$ and $S(t)$ stands for the step function, and $\Lambda^{(1)}$, $\Lambda^{(2)}$ are given previously. Before closing this section, we give an approximate expression of the stable manifold $v_s(t), u_s(t)$, which is the time-reversal image of the unstable manifold:

$$
\begin{aligned}
v_s(t) &= v_{00}(t)+\sigma^2 v_{02}(t) \\
&\quad - S(-t)\mathrm{Re}\left[\frac{4\Lambda^{(1)}}{i\sigma}e^{\frac{2\pi i t_1}{\sigma}}x_2(t)e^{-\frac{2\pi i t}{\sigma}}\right] \\
&\quad - S_+(-t)\mathrm{Re}\left[\frac{4\Lambda^{(2)}}{i\sigma}e^{\frac{2\pi i t_2}{\sigma}}\left\{\left(\frac{T(k-1)^2-(1+k)}{4k(k-1)}\right)x_1(t)+x_2(t)\right\}e^{-\frac{2\pi i t}{\sigma}}\right] \\
u_s(t) &= u_{00}(t)+\sigma\frac{y_1(t)}{2}+\sigma^2 u_{02}(t)+\sigma^3\left(\frac{1}{2}u_{02}'(t)-\frac{1}{24}y_1''(t)\right) \\
&\quad - S(-t)\mathrm{Re}\left[\frac{4\Lambda^{(1)}}{i\sigma}e^{\frac{2\pi i t_1}{\sigma}}\left(y_2(t)+\sigma\frac{y_2'(t)}{2}\right)e^{-\frac{2\pi i t}{\sigma}}\right] \\
&\quad - S_+(-t)\mathrm{Re}\left[\frac{4\Lambda^{(2)}}{i\sigma}e^{\frac{2\pi i t_2}{\sigma}}\left\{\left(\frac{T(k-1)^2-(1+k)}{4k(k-1)}\right)\left(y_1(t)+\sigma\frac{y_1'(t)}{2}\right)\right.\right. \\
&\qquad\qquad \left.\left. +\left(y_2(t)+\sigma\frac{y_2'(t)}{2}\right)\right\}e^{-\frac{2\pi i t}{\sigma}}\right] \qquad (6\cdot6)
\end{aligned}
$$

§7. Comparison with Simulations and Reconnection of Unstable Manifold

7.1. *Comparison with numerical calculations*

Here we compare the analytical solution (6·5) obtained in the previous section with the numerical calculation. (6·5) indicates that, when t exceeds $t = 0$, exponentially growing oscillatory term appears and, when t exceeds $t = 2T$, another exponentially growing term is added. These terms are exponentially small with respect to σ but they grow exponentially and become dominant for sufficiently large t. Thus, added terms describe heteroclinic tangles and a large-amplitude oscillation

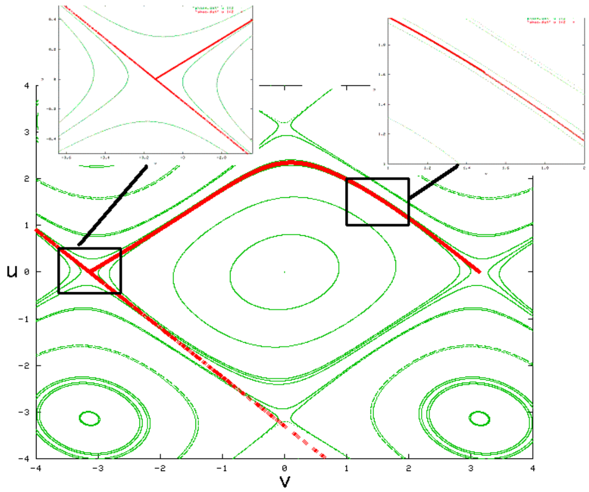

Fig. 4. The analytically constructed unstable manifold (solid line) and numerically portrated phase space. ($\sigma = 0.35, k = 0.85$)

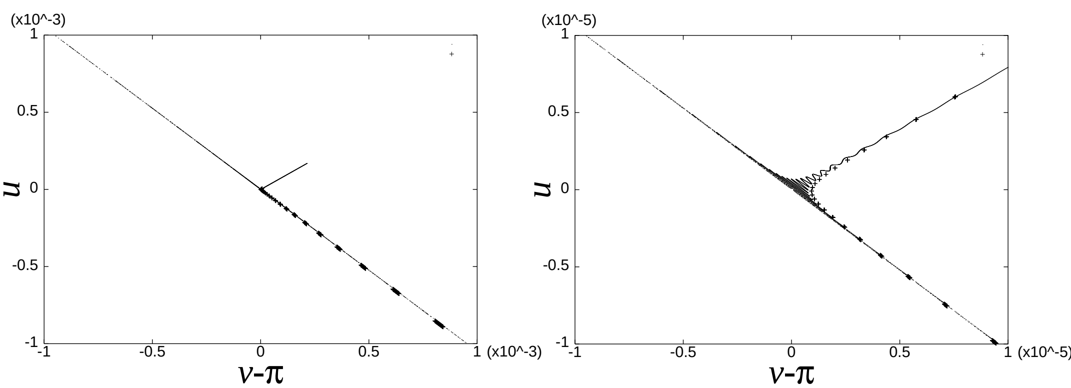

Fig. 5. The analytically constructed unstable manifold (solid line) near $(-\pi, 0)$ and time evolution of the ensemble (plus-mark). ($\sigma = 0.3, k = 0.85$)

appears near $(-\pi, 0)$ can be explained with them. This is indeed the case. Fig. 4 shows the analytically obtained approximate solution of the unstable manifold from $(\pi, 0)$ (solid curve) and numerically portrayed phase space, and Fig. 5 shows the analytically obtained approximate solution of the unstable manifold from $(\pi, 0)$ (solid curve) and the numerically calculated time evolution of an ensemble of 500 points on solid curve (plus-mark) near $(-\pi, 0)$. As shown in Fig. 5, the analytically obtained approximate solution (6·5) (solid curve) well reproduces the numerical results (plus-mark), even the large-amplitude oscillation and stretching of the ensemble near $(-\pi, 0)$. When only the leading order terms of the inner equation are retained, the agreement between the analytical and numerical results is not so good (cf. Fig. 6). We also observe that terms of order σ extends time range in which the solution well

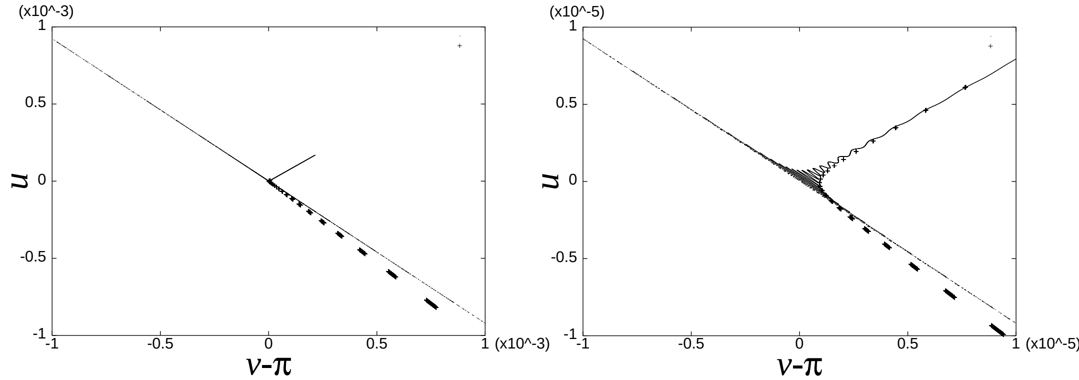

Fig. 6. The analytically constructed unstable manifold (solid line) near $(-\pi, 0)$ and time evolution of the ensemble (plus-mark). Only the leading term is taken into consideration. $(\sigma = 0.3, k = 0.85)$

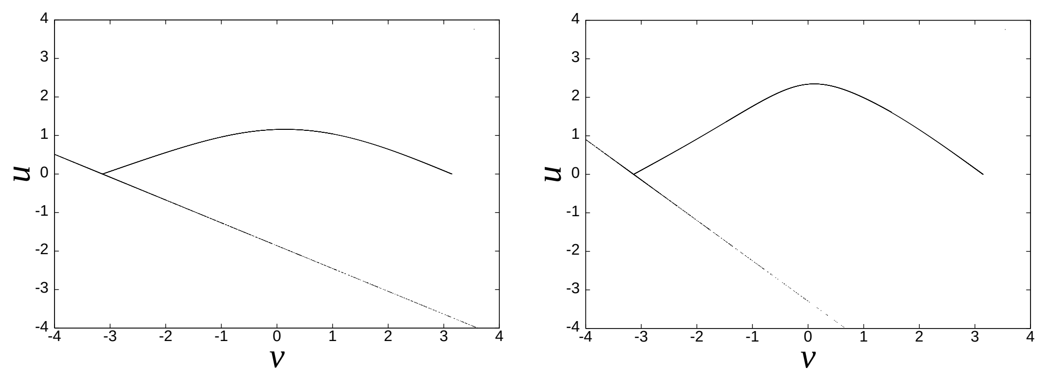

Fig. 7. The analytically constructed unstable manifold. The left figure shows the case of $\sigma = 0.3,\ k = 0.3$ and the right figure shows the case of $\sigma = 0.3,\ k = 0.85$.

represents the unstable manifold and, thus, the terms of order σ of the inner solution are important.

Fig. 7, Fig. 8 show the unstable manifolds when $\sigma = 0.30$ and $k = 0.3,\ 0.85,\ 1.0 - 5.0 \times 10^{-11}$, respectively. The overall structure of the unstable manifold is similar to that of the separatrix except near the fixed point $(-\pi, 0)$. Near $(-\pi, 0)$, the unstable manifold acquires an oscillatory portion and the slope of this portion becomes steeper as k increases 1. The behavior of the slope can be understood from the asymptotic ratio between the additional terms:

$$\lim_{t \to +\infty} \frac{u_1(t)}{v_1(t) + \pi} = -\sqrt{k} - \frac{k}{2}\sigma \ .$$

In Fig. 7, Fig. 8 one can see a small stochastic domain and this ratio represent an unstable direction in the domain near $(-\pi, 0)$.

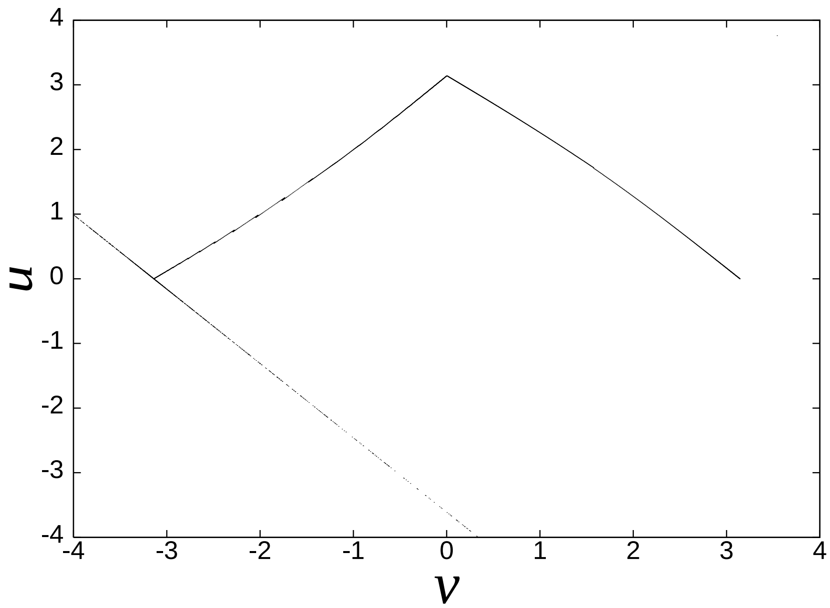

Fig. 8. The analytically constructed unstable manifold for $\sigma = 0.3,\ k = 1.0 - 5.0 \times 10^{-11}$.

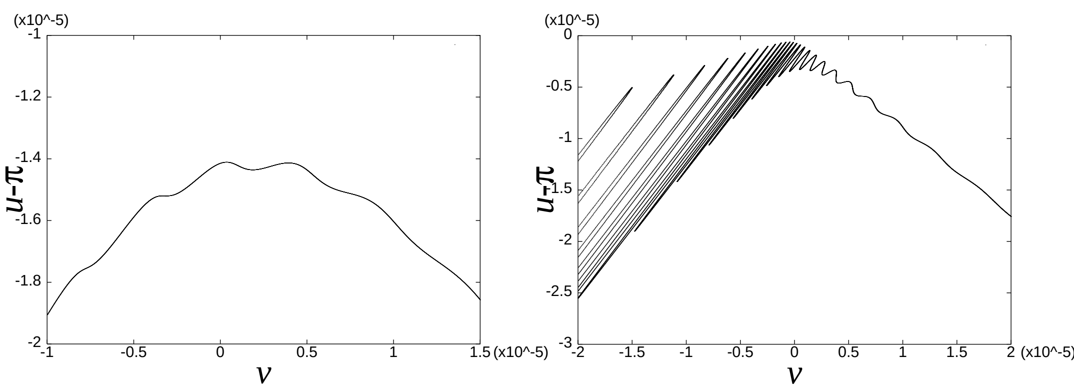

Fig. 9. The analytically constructed unstable manifold near $(0, \pi)$. The left figure shows the case of $\sigma = 0.3,\ k = 1.0 - 5.0 \times 10^{-11}$ and the right figure shows the case of $\sigma = 0.3,\ k = 1.0 - 1.4 \times 10^{-12}$.

7.2. *Reconnection transition*

We remind that the separatrix of the continuous-time limit equation is the unperturbed unstable/stable manifold. As mentioned in introduction (Fig. 1), the separatrix changes its topology depending on the parameter k (reconnection). When $0 < k < 1$, there appears a separatrix connecting $(\pi, 0)$ and $(-\pi, 0)$. As $k \to 1$, the

time when the orbit crosses the u-axis diverges and it approaches the separatrix for $k = 1$ connecting $(\pi, 0)$ with $(0, \pi)$. The corresponding topological change occurs for unstable/stable manifolds, which will be discussed in subthis section. As shown in the previous section, since the approximate stable and unstable manifolds are related with each other by a simple symmetry, it is enough to discuss the topological change of the unstable manifold. It is interesting to see that, when k is very close to unity (i.e., $k = 1.0 - 5.0 \times 10^{-11}$), there appears an additional oscillatory portion in the unstable manifold near $(0, \pi)$ (cf. Fig. 8, Fig. 9). As mentioned before, at $k = 1$, two segments from $(\pi, 0)$ to $(0, \pi)$ and from $(0, \pi)$ to $(-\pi, 0)$ are separatrices of the continuous-limit equation. Thus, the perturbed unstable manifold at $k = 1$ starting from $(\pi, 0)$ should have an oscillatory portion near $(0, \pi)$. The oscillation near $(0, \pi)$ of the unstable manifold shown in Fig. 9 can be considered as the precursor of this oscillation in the unstable manifold at $k = 1$. Origins of these behaviors are summarized as follows.

When k is not very close to 1, the unstable manifold reaches near $(-\pi, 0)$ before additional terms become large. On the other hand, when k is very close to 1, the orbit along the unstable manifold takes very long time to reach u-axis because $v_{00}(T(k)) = 0$ and $\lim_{k \to 1-0} T(k) = \infty$. Moreover, since the solutions at $t = T(k)$ behave:

$$\lim_{k\to 1-0} \begin{pmatrix} v_{00}(T) \\ u_{00}(T) \end{pmatrix} = \begin{pmatrix} 0 \\ \pi \end{pmatrix}$$
$$\lim_{k\to 1-0} \begin{pmatrix} x_2(T) \\ y_2(T) \end{pmatrix} = \begin{pmatrix} \infty \\ -\infty \end{pmatrix},$$

when $k \simeq 1$ the additinal term is not so small compared to the perturbative solution near $(0, \pi)$. Therefore the unstable manifold shows a small oscillation near $(0, \pi)$. We note that the oscillation near $(0, \pi)$ is driven by the first additinal term, on the contrary, the oscillation near $(-\pi, 0)$ is driven by the two additinal terms. We further remark that two oscillatory portions in the unstable manifold for $k \simeq 1$ come from the singularities at $t = t_1, t_1^*$ and the singularities $t = t_2, t_2^*$ goes to infinity as $\lim_{k\to 1-0} \mathrm{Re}\, t_2 = \infty$. In other words, the two sequences of singularities are necessary for the description of the reconnection of unstable/stable manifolds.

In Fig. 9, the unstable manifold for $k = 1.0 - 5.0 \times 10^{-11}$ and $k = 1.0 - 1.4 \times 10^{-12}$ near $(0, \pi)$ are shown. Because of the exponential divergence ($\sim e^{\sqrt{k}t}$) of the additional terms for $t \to \infty$, the approximate unstable manifold (6·5) will not work if $|\epsilon e^{\sqrt{k}t}| \gg 1$, i.e. $t \gg \frac{\pi^2}{k\sigma}$. When k is very close to 1, $T \sim -\frac{1}{2}\ln(1-k) \gg 1$, and, thus, (6·5) is not valid for $t > T$. In this case, (6·5) is applicable for $t \ll T$. Indeed, when $k = 1.0 - 1.4 \times 10^{-12}$, (6·5) shows self crossing curves. Hence, Fig. 9 is drawn by restriction of $t < T$.

§8. Conclusions

With the aid of the ABAO (the asymptotics beyond all orders) method, we have derived analytical approximation of the unstable/stable manifolds, which agree

rather well with those obtained by the numerical iteration. The perturbed unstable manifold starting from $(\pi, 0)$ acquires additional terms which are exponentially small with respect to perturbation parameter σ, but it exponentially grows with respect to time t. Therefore separatrix splitting can be explained by the change of dominant terms among perturbative solution and added terms derived by the ABAO method. And overall approximation shows a highly oscillatory behavior only near the fixed point $(-\pi, 0)$ when k is not close to 1. From the comparison with the numerical calculation, we check that the approximation breaks dowm when time goes to large and that the time regime where the approximation is valid extends thanks to the contribution from the higher order terms with respect to σ. We also mention that when k is very close to 1 we can observe an oscillation near $(0, \pi)$, which is considered to be the precursor of the heteroclinic tangle in the unstable manifold at $k = 1$. In this way, even when the heteroclinic tangle exists, the unstable manifold smoothly changes its topology as the change of the parameter k.

In the systems studied so far such as, the standard map, the Hénon map, and the cubic map, the perturbative solution has only one sequence of singularities in the complex time domain. On the contrary, the Harper map exhibits two sequences of singularities when $k \neq 1$. The interference of the contribution from them might be expected. We concretely write down the interference term and show that it can be negligible. Also we show that it is important, in general, to choose suitable initial condition because incorrect choice of the initial conditions lead to Stokes multipliers(SM) which diverge as $k \to 1$. This also shows that even the perturbative solution does not have a symmetry with respect to u axis. In a higher order approximation, the interference should contribute and such a situation will be discussed elsewhere.

Before closing this section, we summarize the matching conditions and order of the added terms in the ABAO analysis for the following maps (table I).

the standard map[12),21)] : $\begin{pmatrix} v_{n+1} - v_n \\ u_{n+1} - u_n \end{pmatrix} = \begin{pmatrix} \sigma u_{n+1} \\ \sigma \sin v_n \end{pmatrix}$

the Hénon map[14),18)] : $\begin{pmatrix} v_{n+1} - v_n \\ u_{n+1} - u_n \end{pmatrix} = \begin{pmatrix} \sigma u_{n+1} \\ \sigma(v_n^2 - 2v_n) \end{pmatrix}$

the cubic map[19)] : $\begin{pmatrix} v_{n+1} - v_n \\ u_{n+1} - u_n \end{pmatrix} = \begin{pmatrix} \sigma u_{n+1} \\ -4\sigma v_n(v_n - a)(v_n + a) \end{pmatrix}$

as well as the Harper map.

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Map	imaginary part of a nearby singularity	matching between leading order terms	order of added terms
Standard map	$t_c = \frac{i\pi}{2}$	$v_{10}(t_p + \sigma z) \approx z^2 e^{\frac{2\pi i t_c}{\sigma}}$	$\frac{1}{\sigma^2} e^{-\frac{\pi^2}{\sigma}}$
Hénon map	$t_c = \frac{i\pi}{\sqrt{2}}$	$\sigma^2 v_{10}(t_p + \sigma z) \approx z^4 e^{\frac{2\pi i t_c}{\sigma}}$	$\frac{1}{\sigma^6} e^{-\frac{\sqrt{2}\pi^2}{\sigma}}$
Cubic map	$t_c = \frac{i\pi}{4a}$	$\sigma v_{10}(t_p + \sigma z) \approx z^3 e^{\frac{2\pi i t_c}{\sigma}}$	$\frac{1}{\sigma^4} e^{-\frac{\pi^2}{2\sigma a}}$
Harper map	$t_c = \frac{i\pi}{2\sqrt{k}}$	$v_{10}(t_p + \sigma z) \approx z e^{\frac{2\pi i t_c}{\sigma}}$	$\frac{1}{\sigma} e^{-\frac{\pi^2}{\sigma\sqrt{k}}}$

Table I. Order of added terms and matching conditions for several maps. The parameter σ plays a role of the small parameter, t_p stands for the singularity in the upper half plane nearest to the real axis and $t_c = t_p - \mathrm{Re} t_p$ and $v_{10}(t)$ is the additional terms to the outer equation which appears when t exceeds Re t_p.

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Appendix A
—— *Relation between cases of $k > 1$ and $0 < k < 1$* ——

In this section, we consider the case of $k > 1$. Put

$$\tau \equiv kt,\ \widetilde{\sigma} \equiv k\sigma, \quad \widetilde{u}(\tau) \equiv v\left(-\frac{\tau}{k} + \sigma\right),\ \widetilde{v}(\tau) \equiv u\left(-\frac{\tau}{k} + \sigma\right) \tag{A·1}$$

then $\widetilde{v}$, $\widetilde{u}$ are the solution of the Harper map with parameter $1/k$:

$$\begin{aligned} \widetilde{v}(\tau + \widetilde{\sigma}) - \widetilde{v}(\tau) &= -\widetilde{\sigma} \sin \widetilde{u}(\tau) \\ \widetilde{u}(\tau + \widetilde{\sigma}) - \widetilde{u}(\tau) &= \frac{\widetilde{\sigma}}{k} \sin \widetilde{v}(\tau + \widetilde{\sigma}) \end{aligned} \tag{A·2}$$

Appendix B
—— *Contribution from complex conjugate singularity* ——

In the main part of this article, we present details of analytic continuation via the behavior of the solution near t_1. The purpose of this appendix is to prove that the additional term arrising from t_1^* is just the complex conjugate of that from t_1. We present the proof for dominant terms, but we only use a real analyticity of perturbative solution. Therefore, the same discussion can be used for less singular terms. From a real analyticity of (v_{0n}, u_{0n}) and the expansion (2·4) in a main part, (v_{0n}, u_{0n}) has a following expansions near t_1 and t_1^* in the complex time domain.

$$\begin{pmatrix} v_{0i}(t) \\ u_{0i}(t) \end{pmatrix} = \sum_{l=0}^{\infty} \begin{pmatrix} a_l^{(i)}(t - t_1)^l \\ b_l^{(i)}(t - t_1)^l \end{pmatrix} + \sum_{l=0}^{\infty} \begin{pmatrix} \overline{a_l^{(i)}}(t - t_1^*)^l \\ \overline{b_l^{(i)}}(t - t_1^*)^l \end{pmatrix} , \quad (i \geq 1, b_0^{(i)} \neq 0) .$$

Hence, this expansion gives the following asymptotic expansion:

$$\sum_{i=1}^{\infty}\frac{1}{z^i}\begin{pmatrix}a_0^{(i)}\\ b_0^{(i)}\end{pmatrix}+\sum_{i=1}^{\infty}\frac{1}{\widetilde{z}^i}\overline{\begin{pmatrix}a_0^{(i)}\\ b_0^{(i)}\end{pmatrix}} \tag{B·1}$$

where z and $\widetilde{z}$ are defined by $z=(t-t_1)/\sigma$ and $z=(t-t_1^*)/\sigma$. Therefore, it is sufficient to show that the second term generates just the complex conjugate of the first term:

$$\begin{pmatrix}V_{add}(z)\\ U_{add}(z)\end{pmatrix}\equiv-\int_\gamma dpe^{-pz}\sum_{i=1}^{\infty}\frac{p^{i-1}}{(i-1)!}\begin{pmatrix}a_0^{(i)}\\ b_0^{(i)}\end{pmatrix}$$

Let γ' denote symmetric pass of γ with respect to real axis in p plane.By analitically continuing the second term of (B·1) from $\mathrm{Re}\widetilde{z}<0$, $\mathrm{Im}\widetilde{z}>0$ to $\mathrm{Re}\widetilde{z}>0$, $\mathrm{Im}\widetilde{z}>0$ following term is added.

$$\begin{aligned}-\int_{\gamma'} dpe^{-p\widetilde{z}}\sum_{i=1}^{\infty}\frac{p^{i-1}}{(i-1)!}\overline{\begin{pmatrix}a_0^{(i)}\\ b_0^{(i)}\end{pmatrix}}&=-\int_\gamma d\overline{p}e^{-\overline{p}\widetilde{z}}\sum_{i=1}^{\infty}\frac{\overline{p}^{i-1}}{(i-1)!}\overline{\begin{pmatrix}a_0^{(i)}\\ b_0^{(i)}\end{pmatrix}}\\ &=-\int_\gamma dpe^{-p\widetilde{z}^*}\sum_{i=1}^{\infty}\frac{p^{i-1}}{(i-1)!}\begin{pmatrix}a_0^{(i)}\\ b_0^{(i)}\end{pmatrix}\\ &=\overline{\begin{pmatrix}V_{add}(\widetilde{z}^*)\\ U_{add}(\widetilde{z}^*)\end{pmatrix}}\end{aligned}$$

near t_1^*. Let us suppose that $x_{add}(t,\sigma)$ is matched with (B·2) near t_1. Ofcourse, the second term is matched with $\overline{x_{add}(\overline{t},\sigma)}$ and it is the complex conjugate of the first term, $x_{add}(t,\sigma)$, for real t and this fact complete the proof.

Appendix C
—— Borel-transformed linearized inner equation ——

The aim of this appendix is to examine an asymptotic behavior of $(V_A(z),\ U_A(z))$ and $(V_B(z),\ U_B(z))$. The Borel transforms $(\widetilde{V}_A(p),\widetilde{U}_A(p))$ of $(V_A(z),\ U_A(z))$ and $(\widetilde{V}_B(p),\widetilde{U}_B(p))$ of $(V_B(z)-z, U_B(z)-(z+\frac{1}{2}))$ satisfy

$$\begin{aligned}(e^{-p}-1)\widetilde{V}_A(p)&=\widetilde{U}_A(p)*g(p)\\ (1-e^{p})\widetilde{U}_A(p)&=\widetilde{V}_A(p)*f(p)\\ (e^{-p}-1)\widetilde{V}_B(p)&=\widetilde{U}_B(p)*g(p)+\frac{1}{2}g(p)+g'(p)\\ (1-e^{p})\widetilde{U}_B(p)&=\widetilde{V}_B(p)*f(p)+f'(p)\end{aligned} \tag{C·1}$$

where

$$B\left[\frac{e^{iV_{00}(z)}}{z}\right]=f(p),\ B\left[\frac{e^{-iU_{00}(z)}}{z}\right]=g(p)$$

By substituting the power series expansions:

$$\widetilde{V}_A(p) \equiv \sum_{n=0}^{\infty} A_n^V p^n, \quad \widetilde{U}_A(p) \equiv \sum_{n=0}^{\infty} A_n^U p^n; \quad A_0^V = -1, \quad A_0^U = 1$$
$$\widetilde{V}_B(p) \equiv \sum_{n=0}^{\infty} B_n^U p^n, \quad \widetilde{U}_B(p) \equiv \sum_{n=0}^{\infty} B_n^U p^n; \quad B_0^V = 0, \quad B_0^U = \frac{1}{12}$$

into (C·1) and comparing term by term, the coefficients $a_n,\ b_n$ are determined recursively and one has

$$\begin{aligned}
A_{2n-1}^V &= 0, \qquad (\forall n)\\
A_{2n}^V(-1)^n(2\pi)^{2n} &\to -B_4 n \quad (n\to\infty)\\
A_{2n-1}^U(-1)^{n+1}(2\pi)^{2n-1} &\to -B_5 \quad (n\to\infty)\\
A_{2n}^U(-1)^n(2\pi)^{2n} &\to -B_4 n \quad (n\to\infty) && \text{(C·2)}\\
B_{2n-1}^V &= 0, \qquad (\forall n)\\
B_{2n}^V(-1)^n(2\pi)^{2n} &\to B_1(2n+2)(2n+1) - B_2 n \quad (n\to\infty)\\
B_{2n-1}^U(-1)^{n+1}(2\pi)^{2n-1} &\to -B_3 n \quad (n\to\infty)\\
B_{2n}^U(-1)^n(2\pi)^{2n} &\to -B_1(2n+2)(2n+1) - B_2 n \quad (n\to\infty) && \text{(C·3)}
\end{aligned}$$

where $B_1 = 0.01480$, $B_2 = 0.14$, $B_3 = 0.186$, $B_4 = 3.503$, $B_5 = 5.551$. Therefore, the following relations hold:

$$\begin{pmatrix}\widetilde{V}_A(p)\\ \widetilde{U}_A(p)\end{pmatrix} = i\begin{pmatrix}2\pi^3 B_4 f_1^{(R)}(p)\\ 2\pi^3 B_4 f_1^{(R)}(p)\end{pmatrix}$$
$$\begin{pmatrix}\widetilde{V}_B(p)\\ \widetilde{U}_B(p)\end{pmatrix} = i\begin{pmatrix}8\pi^4 B_1 f_2^{(I)}(p) + 2\pi^3 B_2 f_1^{(R)}(p)\\ -8\pi^4 B_1 f_2^{(I)}(p) + 2\pi^3 B_2 f_1^{(R)}(p)\end{pmatrix} + 2\pi^3 i B_3 f_1^{(I)}(p)\begin{pmatrix}0\\1\end{pmatrix}$$

$$-2\pi i e^{2\pi i z}\mathrm{Res}_{\mathrm{p=2\pi i}}\begin{pmatrix}\widetilde{V}_A(p)e^{-pz}\\ \widetilde{U}_A(p)e^{-pz}\end{pmatrix} = 2\pi^3 i B_4 z\begin{pmatrix}1\\1\end{pmatrix}$$
$$-2\pi i e^{2\pi i z}\mathrm{Res}_{\mathrm{p=2\pi i}}\begin{pmatrix}\widetilde{V}_B(p)e^{-pz}\\ \widetilde{U}_B(p)e^{-pz}\end{pmatrix} = 8\pi^4 B_1 z^2\begin{pmatrix}1\\-1\end{pmatrix} + 2\pi^3 i B_2 z\begin{pmatrix}1\\1\end{pmatrix} - 2\pi^3 B_3 z\begin{pmatrix}0\\1\end{pmatrix}$$

Appendix D

—— *Borel-transformed first order solution of inner equation* ——

The aim of this appendix is to examine an asymptotic behavior of the solutions $V_{01}(z),\ U_{01}(z)$. The Borel transforms $\widetilde{V}_1(p), \widetilde{U}_1(p)$ of $V_{01}(z), U_{01}(z) - i\frac{k-1}{4}$ satisfy

$$\begin{aligned}
(e^{-p}-1)\widetilde{V}_1(p) &= i\frac{k-1}{2}g'(p) + i\frac{k-1}{4}g(p) + \widetilde{U}_1(p) * g(p)\\
(1-e^{p})\widetilde{U}_1(p) &= i\frac{k-1}{2}f'(p) + \widetilde{V}_1(p) * f(p) && \text{(D·1)}
\end{aligned}$$

The power series expansions of $\widetilde{V}_1(p),\ \widetilde{U}_1(p)$ is defined by

$$\widetilde{V}_1(p) = -i\frac{kt_1+1}{24} + \sum_{n=1}^{\infty}\widetilde{c}_n p^n\ , \qquad \widetilde{U}_1(p) = i\frac{k(t_1+1)}{24} + \sum_{n=1}^{\infty}\widetilde{d}_n p^n \tag{D·2}$$

(D·1) and (D·2) gives the following form:

$$\widetilde{V}_1(p) = kt_1\widetilde{V}_x(p) + (k-1)\widetilde{V}_y(p) + \widetilde{V}_z(p)\ , \quad \widetilde{U}_1(p) = kt_1\widetilde{U}_x(p) + (k-1)\widetilde{U}_y(p) + \widetilde{U}_z(p) \tag{D·3}$$

where $\widetilde{V}_\alpha(p),\ \widetilde{U}_\alpha(p)$ $(\alpha = x, y, z)$ are independent of k. By substituting (D·3) into (D·1), we get

$$\begin{aligned}
(e^{-p}-1)\widetilde{V}_\alpha(p) &= \widetilde{U}_\alpha(p) * g \\
(1-e^{p})\widetilde{U}_\alpha(p) &= \widetilde{V}_\alpha(p) * f \\
&\quad (\alpha = x, z) \\
(e^{-p}-1)\widetilde{V}_y(p) &= \widetilde{U}_y(p) * g + \frac{1}{4}g(p) + \frac{1}{2}g'(p) \\
(1-e^{p})\widetilde{U}_y(p) &= \widetilde{V}_y(p) * f + \frac{1}{2}f'(p)
\end{aligned} \tag{D·4}$$

By comparing (D·4) with (C·1), we get

$$\begin{aligned}
\widetilde{V}_1(p) &= i\left(\frac{kt_1+1}{24}\widetilde{V}_A(p) + \frac{k-1}{2}\widetilde{V}_B(p)\right) \\
\widetilde{U}_1(p) &= i\left(\frac{kt_1+1}{24}U_A(p) + \frac{k-1}{2}\widetilde{U}_B(p)\right)
\end{aligned} \tag{D·5}$$

This estimation gives (4·33).

Appendix E
—— *Choice of initial time and Stokes Multiplier* ——

In this appendix, the relation between initial time and Stokes multiplier is discussed. Here we restrict our discussion to the case of $(v_{02}(t), u_{02})$, which is used to obtain our final formula. We start from pertarbative solution of $(v_{02}(t), u_{02}(t))$.

$$\begin{aligned}
v_{02}(t) = &-\frac{1}{24}\left[x_1'(t) + x_1(t)\left\{kt - 2\sqrt{k}\frac{\left(\sinh\sqrt{k}t - \sqrt{k}\cosh\sqrt{k}t\right)\left(\cosh\sqrt{k}t - \sqrt{k}\sinh\sqrt{k}t\right)}{(1+k)\cosh^2\sqrt{k}t - 2\sqrt{k}\sinh\sqrt{k}t\cosh\sqrt{k}t}\right\}\right] \\
&+ \frac{A(k)}{24}x_1(t) + \frac{B(k)}{24}x_2(t) \\
u_{02}(t) = &\ \frac{1}{24}\left[2y_1'(t) - y_1(t)\left\{kt - 2\sqrt{k}\frac{\left(\sinh\sqrt{k}t - \sqrt{k}\cosh\sqrt{k}t\right)\left(\cosh\sqrt{k}t - \sqrt{k}\sinh\sqrt{k}t\right)}{(1+k)\cosh^2\sqrt{k}t - 2\sqrt{k}\sinh\sqrt{k}t\cosh\sqrt{k}t}\right\}\right] \\
&+ \frac{A(k)}{24}y_1(t) + \frac{B(k)}{24}y_2(t)
\end{aligned}$$

We note that (A, B) is chosen as $(A, B) = (0, 0)$ in the main part. In this appendix, we put $B = 0$ for $v_{02}(-\infty) = 0,\ u_{02}(-\infty) = 0$ and leave A. Then, the set of second divergent terms in perturabative soluion near t_1 and t_2 are respectively

$$\widetilde{V}_1(p) = i\left(\frac{i\sqrt{k}\pi - 2A(k) + 2}{48}\widetilde{V}_A(p) + \frac{k-1}{2}\widetilde{V}_B(p)\right)$$
$$\widetilde{U}_1(p) = i\left(\frac{i\sqrt{k}\pi - 2A(k) + 2}{48}U_A(p) + \frac{k-1}{2}\widetilde{U}_B(p)\right) \tag{E·1}$$

and

$$\widetilde{V}_1(p) = i\left(\frac{4kT + i\sqrt{k}\pi - 2A(k) - 2}{48}\widetilde{V}_A(p) + \frac{k-1}{2}\widetilde{V}_B(p)\right)$$
$$\widetilde{U}_1(p) = i\left(\frac{-4kT - i\sqrt{k}\pi + 2A(k) + 2}{48}U_A(p) + \frac{k-1}{2}\widetilde{U}_B(p)\right) \tag{E·2}$$

They derive the following Stokes multipliers.

$$\Lambda_1^B = -(k-1)\pi^3 B_2 - \frac{i\sqrt{k}\pi - 2A(k) + 2}{24}\pi^3 B_4$$
$$\Lambda_1^B = -(k-1)\pi^3 B_2 + \frac{4kT + i\sqrt{k}\pi - 2A(k) - 2}{24}\pi^3 B_4 \tag{E·3}$$

where the upper multiplier is for t_1 and the lower one is for t_2. We note that the choice $A(k) = kT = \sqrt{k}\ln\frac{1+\sqrt{k}}{\sqrt{1-k}}$ makes a symmetric perturbative solution with respect to u axis but it also makes the upper multiplier divergent. Therefore this choice should be rejected.

Appendix F
—— *Matching of higher order Terms* ——

In this subsection, Matching between inner and outer equation for higher orders is discussed. The analytic continuation of a inner solution V_{0l} to the next sector is:

$$V_{0l}(z) + e^{-2\pi iz}V_{1l}(z)$$
$$U_{0l}(z) + e^{-2\pi iz}U_{1l}(z)$$

where $V_{1l},\ U_{1l}$ should be determined by the linearized equation. To deal with more general case, we analyze the following linearized inner equation:

$$\Delta V_1(z,\sigma) = \sum_{n=0}^{\infty}\{f_{1n}(z)\sigma^n V_1(z,\sigma) + f_{2n}(z)\sigma^n U_1(z,\sigma)\}$$
$$\Delta U_1(z,\sigma) = \sum_{n=0}^{\infty}\{g_{1n}(z)\sigma^n V_1(z,\sigma) + g_{2n}(z)\sigma^n U_1(z,\sigma)\} \tag{F·1}$$

Next, we seek to find the solution of the type

$$\begin{pmatrix} V_1(z) \\ U_1(z) \end{pmatrix} = \sum_{l=0}^{\infty} \sigma^l \begin{pmatrix} V_{1l}(z) \\ U_{1l}(z) \end{pmatrix} \tag{F·2}$$

Because (F·1) is linear and second difference equation, each $\begin{pmatrix} V_l(z) \\ U_l(z) \end{pmatrix}$ has two undefined constant corresponding to homogeneous solution. It indicates that $V_1,\ U_1$ has the form:

$$\begin{aligned} \begin{pmatrix} V_1(z) \\ U_1(z) \end{pmatrix} &= \sum_{l=0}^{\infty} \sigma^l \begin{pmatrix} V_{1l}(z) \\ U_{1l}(z) \end{pmatrix} \\ &= \sum_{\alpha} \sum_{i=0}^{\infty} \Lambda_\alpha(\sigma) \begin{pmatrix} \sigma^i V_{1i}^{\alpha}(z) \\ \sigma^i U_{1i}^{\alpha}(z) \end{pmatrix} \end{aligned} \tag{F·3}$$

where $\Lambda_\alpha(\sigma)$ can be expanded as a power series of σ and where $\begin{pmatrix} V_\alpha(z) \\ U_\alpha(z) \end{pmatrix}$, $\begin{pmatrix} V_{1i}^{\alpha}(z) \\ U_{1i}^{\alpha}(z) \end{pmatrix}$ are defined by the following rules.

1. $\begin{pmatrix} V_\alpha(z) \\ U_\alpha(z) \end{pmatrix}$ is the independent solutions of $\begin{pmatrix} V_{10}(z) \\ U_{10}(z) \end{pmatrix}$ which satisfy

$$\lim_{|z|\to\infty} z^{-j_\alpha} V_\alpha(z) = 1 \ (\alpha = A, B),\ j_A < j_B$$

   Note that this condition is not imposed on U_α.

2. $\begin{pmatrix} V_{1l}^{\alpha}(z) \\ U_{1l}^{\alpha}(z) \end{pmatrix}$ is the solution when $\begin{pmatrix} V_{10}(z) \\ U_{10}(z) \end{pmatrix} = \begin{pmatrix} V_\alpha(z) \\ U_\alpha(z) \end{pmatrix}$ and $\begin{pmatrix} V_{1i}(z) \\ U_{1i}(z) \end{pmatrix} = \begin{pmatrix} V_{1i}^{\alpha}(z) \\ U_{1i}^{\alpha}(z) \end{pmatrix}$ $(i \geq l-1)$. More concretely, they are inductively defined by the following equations.

$$\Delta \begin{pmatrix} V_j^{\alpha}(z) \\ U_j^{\alpha}(z) \end{pmatrix} = \sum_{j=0}^{l} \begin{pmatrix} f_{1j}(z) V_{l-j}^{\alpha}(z) + f_{2j}(z) U_{l-j}^{\alpha}(z) \\ g_{1j}(z) V_{l-j}^{\alpha}(z) + g_{2j}(z) U_{l-j}^{\alpha}(z) \end{pmatrix}$$

3. {coefficient of z^{j_A} and z^{j_B} in $V_{1l}^{\alpha}(z)$} =0

In (2·2), boundary conditions for v_{ni} , u_{ni} $(n \geq 1)$ and ϵ are unsettled. On the other hands, $\Lambda_l^A,\ \Lambda_l^B$ in (F·3) is determined by analytic continuation of asymptotic expansions. Therefore $\Lambda_l^A,\ \Lambda_l^B$ determine (2·2) by matching the asymptotic expansions with it. Similar to the case of $(V_1^{\alpha}, U_1^{\alpha})$, solutions of $v_{1l}(t),\ v_{1l}(t)$ are

$$\begin{pmatrix} v_{10} \\ u_{10} \end{pmatrix} = c_0^A \begin{pmatrix} v_A(t) \\ u_A(t) \end{pmatrix} + c_0^B \begin{pmatrix} v_B(t) \\ u_B(t) \end{pmatrix}$$

$$\begin{pmatrix} v_{1l}(t) \\ u_{1l}(t) \end{pmatrix} = \sum_{j=0}^{l} \left[c_j^A \begin{pmatrix} v_{1\ l-j}^{A}(t) \\ u_{1\ l-j}^{A}(t) \end{pmatrix} + c_j^B \begin{pmatrix} v_{1\ l-j}^{B}(t) \\ u_{1\ l-j}^{B}(t) \end{pmatrix} \right]$$

.

where $\begin{pmatrix} v_\alpha \\ u_\alpha \end{pmatrix}$ denote the homogeneous solution which satisfy:

$$1 = \lim_{t \to t_1} (t - t_1)^{-j_\alpha} v_\alpha(t)$$

and where $v_{1l}^\alpha(t),\ u_{1l}^\alpha(t),\ (l \geq 1)$ are defined by following 2 conditions.

1. $\begin{pmatrix} v_{1l}^\alpha \\ u_{1l}^\alpha \end{pmatrix}$ is the solution when $\begin{pmatrix} v_{10} \\ u_{10} \end{pmatrix} = \begin{pmatrix} v_\alpha \\ u_\alpha \end{pmatrix}$ and $\begin{pmatrix} v_{1i} \\ u_{1i} \end{pmatrix} = \begin{pmatrix} v_{1i}^\alpha \\ u_{1i}^\alpha \end{pmatrix}$ $(0 < i \leq l-1)$
2. {coefficient of $(t - t_1)^{j_\alpha}$ in v_{1l}^α} $=0$

The matching condition is:

$$e^{-2\pi i z} \sum_{l=0}^{\infty} \sigma^l V_{1l}(z) = E(t) \sum_{i=0}^{\infty} \sigma^i v_{1i}(t) \tag{F·4}$$

As a result, we get

$$E(t) = \sigma^{-j_B} e^{-2\pi i z}$$
$$\sum_{l=0}^{\infty} \sigma^l \begin{pmatrix} V_{1l}(z) \\ U_{1l}(z) \end{pmatrix} = \sigma^{-j_B} \sum_{i=0}^{\infty} \sigma^i \begin{pmatrix} v_{1i}(t) \\ u_{1i}(t) \end{pmatrix}$$

and the second equality leads

$$\begin{aligned} &c_j^A = 0, (j < j_B - j_A) \\ &c_j^A = \Lambda_{j-j_A-j_B}^A, \quad (j \geq j_B - j_A) \\ &c_j^B = \Lambda_j^B. \end{aligned}$$

The rest of this appendix is devoted to present a concrete calculation for the Harper map. The Harper map has the following values.

$$j_A = -1,\ j_B = 1$$

It indicates the following relation.

$$E(t) = \frac{1}{\sigma} e^{-2\pi i z} = \frac{1}{\sigma} e^{\frac{2\pi i t_1}{\sigma}}$$
$$\sum_{l=0}^{\infty} \sigma^l \begin{pmatrix} V_{1l}(z) \\ U_{1l}(z) \end{pmatrix} = \frac{1}{\sigma} \sum_{i=0}^{\infty} \sigma^i \begin{pmatrix} v_{1i}(t) \\ u_{1i}(t) \end{pmatrix}$$

As noted above, the coefficient $\Lambda_\alpha(\sigma) \equiv \sum_{n=0}^{\infty} \sigma_n \Lambda_n^\alpha$ determine whole solutions in (2·2). Let us concretely write down the first two terms:

$$\begin{bmatrix} -\displaystyle\int_\gamma dp\, e^{-pz} \{\widetilde{V}_0(p) + \sigma \widetilde{V}_1(p)\} \\ -\displaystyle\int_\gamma dp\, e^{-pz} \{\widetilde{U}_0(p) + \sigma \widetilde{U}_1(p)\} \end{bmatrix}$$

$$\approx\left(\Lambda_B \left[\begin{array}{c} (V_{10}^B(z) + V_{11}^B(z)\sigma + \cdots) \\ (U_{10}^B(z) + U_{11}^B(z)\sigma + \cdots) \end{array} \right] + \Lambda_A \left[\begin{array}{c} (V_{10}^A(z) + V_{11}^A(z)\sigma + \cdots) \\ (U_{10}^A(z) + U_{11}^A(z)\sigma + \cdots) \end{array} \right] \right) e^{-2\pi i z}$$
$$\approx\left[\begin{array}{l} \Lambda_0^B z + \sigma(\Lambda_1^B z + \frac{k-1}{6}\Lambda_0^B z^2) \\ \Lambda_0^B z + \sigma\left(\Lambda_1^B z - \frac{k-1}{6}\Lambda_0^B (z^2+z)\right) \end{array} \right] e^{-2\pi i z}$$

where the contour integral is evaluated with the aid of (4·27) and (4·38). This leads to the following relations

$$\begin{aligned} \Lambda_0^B &= i4\pi^3 A_1 = 24\pi^4 i B_1 = 6\pi^3 i B_3 \\ \Lambda_1^B &= -(k-1)\pi^3 B_2 - \frac{kt_1+1}{12}\pi^3 B_4 \end{aligned} \tag{F·5}$$

The above relations imply two linear relations among A_1, B_1 and B_3, which are satisfied by the present numerical estimations rather well:

$$\frac{A_1}{6\pi B_1} = 0.99998 \simeq 1 \qquad \frac{2A_1}{3B_3} = 0.99997 \simeq 1 \ .$$

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